MA676: Bayesian Methods – Exercise Sheet 1

1. Suppose that $X_1, \ldots, X_n$ is a random sample from the distribution with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Suppose also that the value of the parameter θ is unknown ($\theta > 0$) and that the prior distribution of θ is a gamma distribution with parameters α and β ($\alpha > 0$ and $\beta > 0$). Determine the posterior distribution of θ and hence obtain the Bayes estimator of θ under a squared error loss function.

2. A Bayes estimator is required for θ under the loss function

$$L(a, \theta) = e^{c(a-\theta)} - c(a - \theta) - 1,$$

   where c is a positive constant. As the constant c varies, the loss function varies from very asymmetric to almost symmetric. This is called the LINEX (LINear-EXponential) loss.

   By minimising the expected loss, show that the Bayes estimator is

$$\hat{\theta} = \frac{-1}{c} \log E\left(e^{-c\theta}|\mathbf{x}\right),$$

   where the expectation is under the posterior distribution $\pi(\theta|\mathbf{x})$.

   Suppose that $X_1, \ldots, X_n$ is a random sample from the normal distribution $N(\theta,\ \sigma^2)$ where σ^2 is known. A priori θ follows the normal distribution with mean μ and variance τ^2 where both μ and τ^2 are known. Find the Bayes estimators of θ under the above loss function and under the squared error loss. Compare the two estimators. **Hint:** Recall the definition and the expression for the moment generating function of a normal distribution.

3. A random variable X has a gamma distribution gamma(m, β) with pdf

$$f(x) = \frac{\beta^m}{\Gamma(m)} x^{m-1} \exp(-\beta x) \qquad x > 0.$$

   Show that if $Y = 1/X$ then Y has p.d.f.

$$f(y) = \frac{\beta^m}{\Gamma(m)} \frac{1}{y^{m+1}} \exp(-\beta/y) \qquad y > 0.$$

   This is the inverse gamma distribution.

   A particular measuring device has normally distributed error with mean zero and unknown variance σ^2. In an experiment to estimate σ^2, n independent evaluations of this error are obtained.

If the prior distribution for σ^2 is inverse gamma with parameters β and m, show that the posterior distribution is also inverse gamma, with parameters β^* and m^*, and derive expressions for β^* and m^*. Show that the predictive distribution for the error, Z, of a further observation made by this device has p.d.f.

$$f(z) \;\propto\; \left(1+\frac{z^2}{2\beta^*}\right)^{-m^*-\frac{1}{2}} \qquad z \in \mathbb{R}.$$

4. Assume $Y_1, Y_2, \ldots, Y_n$ are independent observations which have the distribution $Y_i \sim N(\beta x_i\,, \sigma^2)$, $i = 1, 2, \ldots, n$, where the x_is and σ^2 are known constants, and β is an unknown parameter, which has a normal prior distribution with mean β_0 and variance σ_0^2, where β_0 and σ_0^2 are known constants.

   (i) Derive the posterior distribution of β.

   (ii) Show that the mean of the posterior distribution is a weighted average of the prior mean β_0, and the maximum likelihood estimator of β.

   (iii) Find the limit of the posterior distribution as $\sigma_0^2 \to \infty$, and discuss the result.

   (iv) How would you predict a future observation from the population $N(\beta x_{n+1}\,, \sigma^2)$, where x_{n+1} is known?

5. Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of independent, identically distributed random variables with probability density function

$$f(y|\lambda) \;=\; \begin{cases} \lambda e^{-\lambda y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

   where λ is an unknown, positive parameter with a gamma(m, β) prior distribution (see above).

   (i) Show that the posterior distribution of λ given $Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n$ is gamma$(n+m, \beta+t)$ where $t = \sum_{i=1}^{n} y_i$.

   (ii) Show that the (predictive) density of Y_{n+1} given the n observations $Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n$ is

$$\pi(y_{n+1}|y_1, \ldots, y_n) \;=\; \frac{(n+m)(\beta+t)^{n+m}}{(y_{n+1}+\beta+t)^{n+m+1}}.$$

   (iii) Find the joint (predictive) density of Y_{n+1} and Y_{n+2} given $Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n$.