MA676: Bayesian Methods – Exercise Sheet 2

1. Let $X_1, X_2, \ldots, X_6$ be a sequence of independent, identically distributed Bernoulli random variables with parameter θ, and suppose that $x_1 = x_2 = x_3 = x_4 = x_5 = 1$ and $x_6 = 0$.

   Derive the posterior model probabilities for Model $0 : \theta = \frac{1}{2}$ and Model $1 : \theta > \frac{1}{2}$, assuming the following prior distributions:

   (a) $P(M_0) = 0.5$, $P(M_1) = 0.5$, $\pi_1(\theta) = 2; \quad \theta \in \left(\frac{1}{2}, 1\right)$.

   (b) $P(M_0) = 0.8$, $P(M_1) = 0.2$, $\pi_1(\theta) = 8(1-\theta); \quad \theta \in \left(\frac{1}{2}, 1\right)$.

   (c) $P(M_0) = 0.2$, $P(M_1) = 0.8$, $\pi_1(\theta) = 48\left(\theta - \frac{1}{2}\right)(1-\theta); \quad \theta \in \left(\frac{1}{2}, 1\right)$.

2. Suppose that:

$$M_0 : X_1, X_2, \ldots, X_n | \theta_0 \sim f_0(x|\theta_1) = \theta_0(1-\theta_0)^x, \quad x = 0, 1, \ldots.$$

$$M_1 : X_1, X_2, \ldots, X_n | \theta_1 \sim f_1(x|\theta_1) = e^{-\theta_1}\theta_1^x / x!, \quad x = 0, 1, \ldots.$$

   Suppose that θ_0 and θ_1 are both unknown. Assume that $\pi_0(\theta_0)$ is the beta distribution with parameters α_0 and β_0 and $\pi_1(\theta_1)$ is the Gamma distribution with parameters α_1 and β_1. Compute the (prior) predictive means under the two models. Obtain the Bayes factor. Hence study the dependence of the Bayes factor on prior data combinations. Calculate numerical values for $n = 2$ and for two data sets $x_1 = x_2 = 0$ and $x_1 = x_2 = 2$ and two sets of prior parameters $\alpha_0 = 1, \beta_0 = 2$, $\alpha_1 = 2, \beta_1 = 1$ and $\alpha_0 = 30, \beta_0 = 60$, $\alpha_1 = 60, \beta_1 = 30$. Write a computer program (Splus preferable) to calculate the Bayes factor for given values of $x_1, \ldots, x_n$ and the parameters.

3. Suppose that $X_1, \ldots, X_n$ is a sample from the negative binomial distribution which has the probability mass function,

$$f(x|r,\theta) = \binom{r+x-1}{x}\theta^r(1-\theta)^x, \quad x = 0, 1, \ldots; \; 0 < \theta < 1,$$

   where $r > 0$ is a known integer. Suppose also that a-priori θ has a beta(α, β) distribution with the pdf,

$$\pi(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \; 0 < \theta < 1.$$

   (a) Find the posterior distribution of θ.

   (b) Suppose further that $r = 2, n = 1$, and we observe that $x_1 = 1$. Of the two hypotheses $H_1 : \theta \leq 0.5$ and $H_2 : \theta > 0.5$, which has greater posterior probability under the uniform prior.

   (c) What is the Bayes factor in favor of H_2? Does it suggest strong evidence in favor of this hypothesis?

4. In an experiment to compare two measuring devices n_1 objects are measured with the first device, the mearsurements errors being recorded as $x_1, \ldots, x_{n_1}$, and n_2 objects are measured with the second device, the mearsurements errors being recorded as $x_{n_1+1}, \ldots, x_{n_1+n_2}$. It is assumed that measurement errors are normally distributed with zero mean. Two models are proposed.

   The first model assumes that the measuring devices are identical and variance of both devices is ϕ (unknown). That is $x_1, \ldots, x_{n_1+n_2}$ is a sample of i.i.d. observations from $N(0, \phi)$.

   The second model allows for a difference between the variances, with $x_1, \ldots, x_{n_1}$ being a sample of i.i.d. observations from $N(0, \phi_1)$ and $x_{n_1+1}, \ldots, x_{n_1+n_2}$ from $N(0, \phi_2)$.

   Assume that the prior distributions for ϕ, ϕ_1 and ϕ_2 are all inverse gamma distributions

$$\pi(\phi) \quad = \quad \frac{\beta^m}{\Gamma(m)} \frac{1}{\phi^{m+1}} \exp(-\beta/\phi), \qquad \phi > 0$$

   with the same m and β for each case.

   Obtain the Bayes factor for comparing the models.

5. Suppose that $Y_1, \ldots, Y_n$ are independently distributed as $N(\beta x_i, \sigma^2)$ where σ^2 and x_i's are known constants. Assume that β follows $N(0, \tau^2)$ a-priori. Find the Bayes factor where one model corresponds to $\beta = 0$ and the other model does not specify any particular value of β. Hence, show that the classical test statistic for testing $H_0 : \beta = 0$ can be obtained as a function of the Bayes factor (in the limit as $\tau^2 \to \infty$).