Chapter 2

Probability and Probability Distributions

2.1 Introduction

Most of us have an idea about probability from games of chance, from the lottery and from general statements about the likelihood of a particular event occurring. The probability of it raining in Southampton tomorrow might be given or the chance that a particular team will win a given match. It will be necessary to clarify ideas about probability a little in order to tackle the kind of problems that we shall meet later, but you will not be required to delve very deeply into the theory of probability.

Firstly, we shall identify a probability of zero with some event which cannot happen and a probability of unity for something which is certain to occur. All other probabilities will be between zero and one and will reflect the "chance" of an event occurring. For a repeatable event, the probability may be interpreted as the proportion of times the event will occur in the "long run". For other kinds of event, probability may be interpreted as a measure of subjective belief reflecting the likelihood of the event occuring.

In this chapter, we consider tightly controlled situations, where it is possible to calculate probabilities precisely. More generally, we cannot know probabilities precisely, but we can use observed data to learn about probabilities – this is statistical inference and is the subject of later chapters.

For example, suppose that electronic resistors of a similar appearance are either 5 ohms or 10 ohms, and we put 100 of the 5 ohm resistors in a box together with 50 of the 10 ohm resistors. A resistor is then chosen from the box. What is the probability that it is a 5 ohm resistor?

It is not immediately possible to answer this question since we are not told enough about the conduct of the experiment. If we are told that the 150 resistors are shaken up in the box and that the resistor is chosen “at random” from the box, then we can argue that each of the resistors has an equal probability of being selected. Since there are now 150 resistors in total and they are all equally likely to be chosen, the probability that a 5 ohm resistor is chosen will be given by $100/150 = 2/3$. Thus the probability of choosing a 5 ohm resistor is formally given by

$$P(\text{5 ohm resistor being chosen}) = \frac{\text{Number of 5 ohm resistors in the box}}{\text{Total number of resistors in the box}} = \frac{2}{3}$$

Similarly, $P(\text{10 ohm resistor being chosen}) = 50/150 = 1/3$

Suppose now we take out a second resistor at random from those left in the box. What is the probability of getting two 5 ohm resistors?

To answer this, consider the experiment in two stages.

(a) Select the first resistor. The probability of a 5 ohm resistor is $2/3$.

(b) Now, assuming that a 5 ohm resistor has been selected, choose the second resistor. There are only 149 resistors left and 99 of them are 5 ohm resistors, so the probability of a 5 ohm resistor being selected is $99/149$.

The probability of getting two 5 ohm resistors is now given by

$$\frac{2}{3} \times \frac{99}{149} = \frac{66}{149} = 0.443.$$

Similarly, the probability of two 10 ohm resistors is

$$\frac{1}{3} \times \frac{49}{149} = \frac{49}{447} = 0.110.$$

The other possibility is that we choose one 5 ohm and one 10 ohm resistor. The probability of this is slightly more involved since we could choose the 5 ohm first and then the 10 ohm resistor or the 10 ohm first and then the 5 ohm resistor. The probability is given by

$$\left(\frac{2}{3} \times \frac{50}{149}\right) + \left(\frac{1}{3} \times \frac{100}{149}\right) = \frac{200}{447} = 0.447.$$

Note that $0.443 + 0.110 + 0.447 = 1$, *i.e.* $P(\text{two 5 ohm}) + P(\text{two 10 ohm}) + P(\text{one of each}) = 1$. Since these are the only possible outcomes, the probabilities must sum to 1.

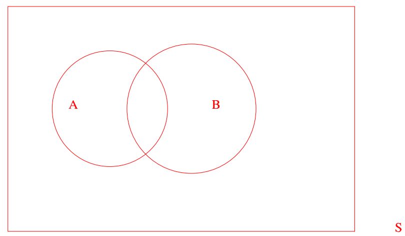
The above example illustrates sampling without replacement, in that the first selected resistor was not replaced in the box before the second was selected.

If we had decided to replace the first resistor, whatever its resistance, before selecting the second, then the probabilities of two 5 ohm, two 10 ohm or one of each would be given by

$$\begin{aligned} P(\text{two 5 ohm}) &= \tfrac{2}{3} \times \tfrac{2}{3} = \tfrac{4}{9} = 0.444 \\ P(\text{two 10 ohm}) &= \tfrac{1}{3} \times \tfrac{1}{3} = \tfrac{1}{9} = 0.111 \\ P(\text{one of each}) &= \left(\tfrac{2}{3} \times \tfrac{1}{3}\right) + \left(\tfrac{1}{3} \times \tfrac{2}{3}\right) = \tfrac{4}{9} = 0.444. \end{aligned}$$

These probabilities for the with replacement scheme are slightly different but, as before, these three situations include all possibilities so the three probabilities must sum to 1.

Notice that we have multiplied probabilities together where considering events occurring together, such as choosing a 5 ohm resistor on the first selection **and** a 5 ohm on the second selection. We have added together probabilities when a situation could arise in two different ways, such as "one of each" could be obtained either as a 5 ohm selected first and a 10 ohm second **or** a 10 ohm selected first and a 5 ohm resistor selected second.



More generally, if we have events A and B, then

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \text{ and } B)$$

and

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B \text{ given that } A \text{ has occured}).$$

If the occurence, or otherwise, of A does not affect the probability of B, then we say that A and B are **independent** events, and we can write $P(B \text{ given that } A \text{ has occured}) = P(B)$. In this case

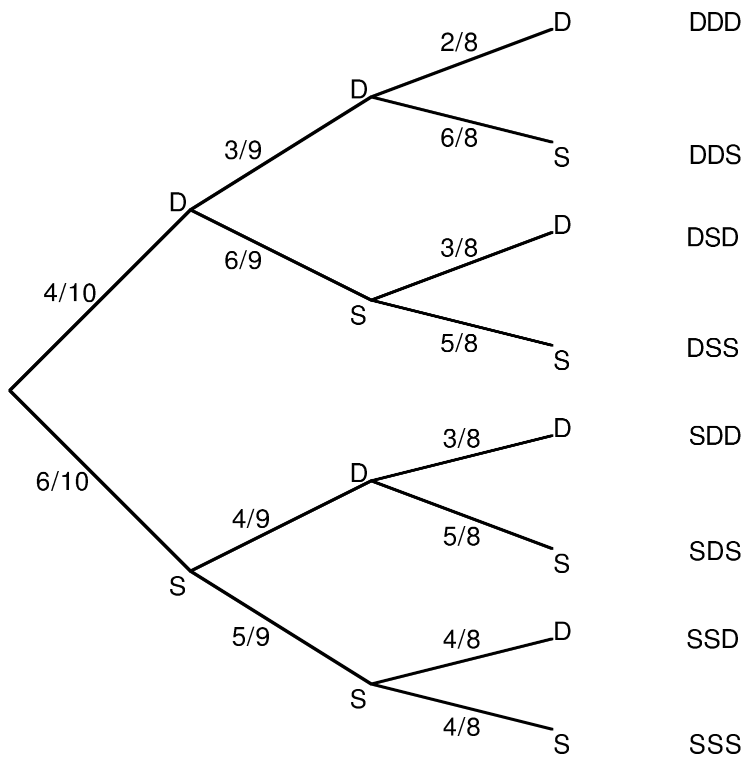
$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B).$$

These simple multiplication and addition rules for probabilities are very important for most problems. The rest of this Section is devoted to a series of examples illustrating the calculation of probabilities using these rules. We shall consider conditional probability in more detail in Section 2.2.

♡ **Example** 2.1. Ten items are available and 4 are defective and 6 are satisfactory. A random sample of 3 items is taken from these 10, what is the probability that exactly one is defective?

One way to tackle a problem like this is to construct a probability tree diagram to see what is going on. Consider selecting one item at a time until all three are selected and illustrate the results and the associated probabilities in each case. (D = defective, S = satisfactory).

So the probability for DDD will be: $\frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$. All the remaining probabilities can be found similarly.



There are eight possible sequences with the probabilities as given in the table above. Note that the sequences DSS, SDS and SSD all have one defective, so the probability of obtaining one defective is given by

$$\left(\frac{4}{10}\times\frac{6}{9}\times\frac{5}{8}\right)+\left(\frac{6}{10}\times\frac{4}{9}\times\frac{5}{8}\right)+\left(\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\right)=3\times\frac{6\times5\times4}{10\times9\times8}=\frac{1}{2}$$

Similarly, the probability of two defectives is

$$\begin{aligned}P(\text{DDS})+P(\text{DSD})+P(\text{SDD}) &= \left(\tfrac{4}{10}\times\tfrac{3}{9}\times\tfrac{6}{8}\right)+\left(\tfrac{4}{10}\times\tfrac{6}{9}\times\tfrac{3}{8}\right)+\left(\tfrac{6}{10}\times\tfrac{4}{9}\times\tfrac{3}{8}\right)\\ &= 3\times\tfrac{6\times4\times3}{10\times9\times8}\\ &= \tfrac{3}{10},\end{aligned}$$

the probability of no defectives is

$$P(\text{SSS})=\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}=\frac{1}{6}$$

and the probability of three defectives is

$$P(\text{SSS})=\frac{4}{10}\times\frac{3}{9}\times\frac{2}{8}=\frac{1}{30}.$$

Note that these four probabilities must sum to 1, *i.e.*

$$P(0\text{ defectives})+P(1\text{ defective})+P(2\text{ defectives})+P(3\text{ defectives})=\frac{1}{6}+\frac{1}{2}+\frac{3}{10}+\frac{1}{30}=1.$$

In fact, we can calculate these probabilities without constructing a probability tree diagram. To do this, we need to know something about **combinations**.

Suppose that we have n items from which we select r without replacement. The order in which the items are selected does not matter, just which r items comprise the final selection. We denote by $\binom{n}{r}$ the number of such distinct combinations of r items which can be selected. It can be shown that

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n \times (n-1) \times \cdots \times (n-r+1)}{1 \times 2 \times \cdots \times r}$$

where $a!$ ("a factorial") is defined to be $a! = a \times (a-1) \times (a-2) \times \cdots \times 3 \times 2 \times 1$. Hence, in particular

$$\begin{aligned} \binom{n}{1} &= n \\ \binom{n}{2} &= \tfrac{n(n-1)}{2} \\ \binom{n}{3} &= \tfrac{n(n-1)(n-2)}{6}. \end{aligned}$$

As we have a total of 10 items, 4 defective and 6 satisfactory. The number of possible ways of selecting 3 items from 10 is

$$\binom{10}{3} = \frac{10 \times 9 \times 8}{6} = 120$$

In order to get one defective and two satisfactory in the sample, the defective must be selected from one of the four defectives and the two satisfactory ones from the six which are satisfactory. Therefore, the number of different selections of one defective and two satisfactory is

$$\binom{4}{1} \times \binom{6}{2} = 4 \times \frac{6 \times 5}{2} = 60$$

Therefore, the probability of choosing one defective in the sample of three is

$$\begin{aligned} P(\text{one defective}) &= \frac{\text{Number of ways of choosing 1 defective and 2 satisfactory}}{\text{Number of ways of choosing 3 items}} \\ &= \tfrac{\binom{4}{1} \times \binom{6}{2}}{\binom{10}{3}} \\ &= \tfrac{60}{120} \\ &= \tfrac{1}{2}. \end{aligned}$$

Similarly

$$\begin{aligned} P(\text{two defectives}) &= \tfrac{\binom{4}{2} \times \binom{6}{1}}{\binom{10}{3}} \\ &= \tfrac{6 \times 6}{120} \\ &= \tfrac{3}{10}. \end{aligned}$$

Either method will produce the answer, but the tree-diagram method can get a bit cumbersome with larger problems.

♡ **Example** 2.2. **The National Lottery** In the National Lottery, the winning ticket has six numbers from 1 to 49 exactly matching those on the balls drawn on a Wednesday or Saturday evening. The 'experiment' consists of drawing the balls. The 'randomness', the equal probability of any set of six numbers being drawn, is ensured by the Lottery machine, which mixes the balls during the selection process.

The probability associated with the winning selection is given by

$$P(\text{Jackpot}) = \frac{\text{Number of winning selections}}{\text{Number of possible selections}}$$

The total number of possible selections is given by

$$\binom{49}{6} = \frac{49 \times 48 \times 47 \times 46 \times 45 \times 44}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 13\,983\,816$$

(*i.e.* nearly 14 million). Since there is only one winning selection, the probability of matching the jackpot sequence is $1/13\,983\,816 = 0.0000000715$.

Other prizes are given for fewer matches. The corresponding probabilities can be evaluated as follows:

$$\begin{aligned} P(5\text{ matches}) &= \frac{\text{Number of selections with 5 matches}}{\text{Number of possible selections}} \\ &= \frac{\binom{6}{5}\times\binom{43}{1}}{\binom{49}{6}} \\ &= \frac{\frac{6!}{5!1!}\times\frac{43!}{1!42!}}{13\,983\,816} \\ &= \frac{6\times 43}{13\,983\,816} \\ &= 0.00001845 \\ &\approx \frac{1}{54\,200} \end{aligned}$$

Similarly,

$$\begin{aligned} P(4\text{ matches}) &= \frac{\binom{6}{4}\times\binom{43}{2}}{\binom{49}{6}} \\ &= \frac{15\times 903}{13\,983\,816} \\ &= 0.0009686 \\ &\approx \frac{1}{1\,032} \end{aligned}$$

$$\begin{aligned} P(3\text{ matches}) &= \frac{\binom{6}{3}\times\binom{43}{3}}{\binom{49}{6}} \\ &= \frac{20\times 12\,341}{13\,983\,816} \\ &= 0.01765 \\ &\approx \frac{1}{57} \end{aligned}$$

There is one other way of winning, using the bonus ball. Matching five of the selected six balls plus matching the bonus ball gives a share in a prize substantially less than the jackpot. The probability of this is given by

$$\begin{aligned} P(\text{Matching 5 and the bonus ball}) &= \frac{\text{Number of selections of this type}}{\text{Number of possible selections}} \\ &= \frac{6}{\binom{49}{6}} \\ &= 0.000000429 \\ &\approx \frac{1}{2\,331\,000} \end{aligned}$$

Adding all these probabilities of winning some kind of prize together gives

$$P(\text{Winning}) = 0.0188 \approx \frac{1}{53}$$

so that a player buying one ticket each week would expect to win a prize about once a year. Without further information, it is not possible to work out the expected return on this kind of investment since this involves the amounts of the prizes as well as the probabilities of winning. In the National Lottery, the prize money, (except for the $10 prize), depends on the number of winners and the number of tickets sold.

One of the most common applications of probability calculations in Engineering is in evaluating reliability. The remaining examples focus on this area.

♡ **Example** 2.3. If a communications satellite is to be launched and positioned in space to receive and transmit telephone and data transmissions, various stages of the process are said to succeed or fail with certain probabilities. For example, it may be that the launch will be successful with a probability of 0.9. The reliability, which is the probability that it works, is therefore 0.9 or 90%. Obviously, the probability that the launch will fail is $1 - 0.9 = 0.1$.

Suppose such a satellite has a successful launch with a probability of 0.9 and after launch, the satellite is to be positioned in a suitable orbit with a probability of 0.8. Small retro-rockets on the satellite can then be used to adjust the position, if this is not initially correct, and the probability of success here is 0.5. Once in position, the solar powered batteries are expected to last at least a year with probability 0.7. What is the probability that a satellite due to be launched will still be working in a year's time?

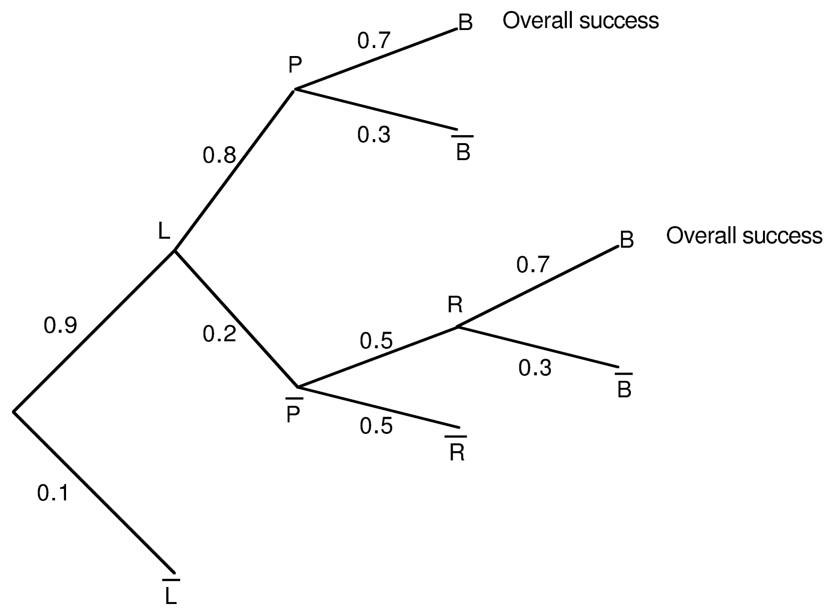
In order to work out this probability, it is necessary to assume that all the different ways of failing are acting independently of each other. This might not be so, of course. if the batteries were used to power the retro-rockets. A simple tree-diagram helps here.

Let L represent a successful launch and $\overline{\text{L}}$ represent a failure, with P, R and B representing successful position, retro-rocket adjustment and battery life, respectively.

The probability of overall success is given by

$$\begin{aligned} (0.9 \times 0.8 \times 0.7) + (0.9 \times 0.2 \times 0.5 \times 0.7) &= 0.504 + 0.063 \\ &= 0.567. \end{aligned}$$

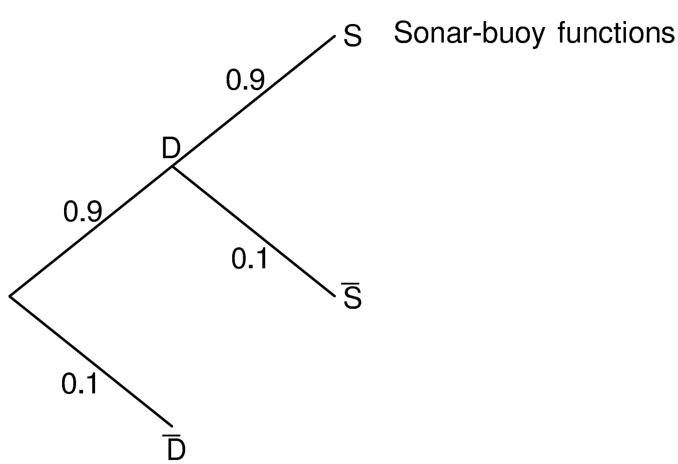
The overall reliability is 56.7%.



Note that whenever a system is affected by a series of different reasons for failure, the overall reliability of the system is reduced. Another example of this follows.

♡ **Example** 2.4. A sonar-buoy dropped from an aircraft to monitor submarines has to deploy its antennae and switch on its transmitter to send signals. If the reliabilities of both the deploying mechanism and the transmitter switch are 90%, what is the reliability of the sonar-buoy?

The following simple diagram will help here.



$$\begin{aligned} P(\text{sonar-buoy functions}) &= P(\text{deploys antennae}) \times P(\text{switch works}) \\ &= 0.9 \times 0.9 \\ &= 0.81 \end{aligned}$$

Therefore the reliability of sonar-buoy is 81%. Although 9 out of 10 of the deploying mechanisms work and 9 out of 10 of the switches work, only 4 out of 5 sonar-buoys work.

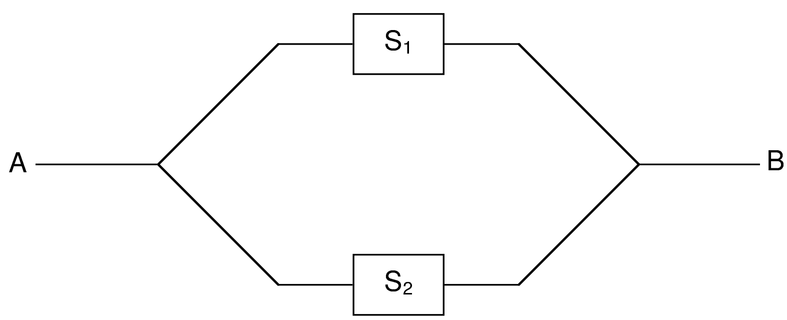
To achieve a 90% reliability for the buoys, we need to have individual reliabilities of $\sqrt{0.9} = 0.9487$ for the switches and deployment mechanisms.

The more components which are required to function to make a system work, the lower the overall reliability. For example, a set of four elements, each with reliability 90%, produces a system with reliability $0.9^4 = 65.6\%$.

Standby redundancy can be used to improve the reliability of a system. It is common practice, when high reliability is required to introduce parallel systems which 'cut-in' if the initial system fails. Some aircraft systems can have as many as three parallel systems, any one of which would be sufficient to fly the plane safely.

♡ **Example** 2.5. Suppose a system consists of two independent switches S_1 and S_2, each with reliability 90% and is arranged so that the system operates if either of the switches, S_1 or S_2, operates. What is the reliability of this system?

This can be represented as below.



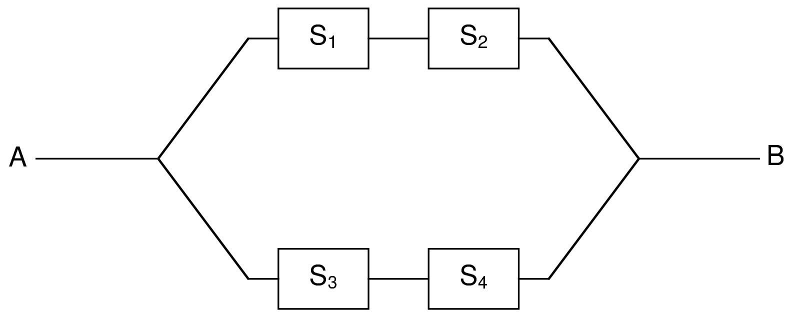
This diagram indicates that the system operates if there is a link from A to B created by the switches operating. The system operates if either or both of the switches are operating. In other words, the system fails only if both switches fail.

$$\begin{aligned} P(\text{system fails}) &= P(\text{switch } S_1 \text{ fails}) \times P(\text{switch } S_2 \text{ fails}) \\ &= 0.1 \times 0.1 \\ &= 0.01 \end{aligned}$$

Therefore, the reliability of the system is 99%.

By introducing a 'spare' switch, the reliability has increased from 90% to 99%, a substantial gain for the potentially small cost of an extra switch.

♡ **Example** 2.6. Systems can be made up of components in 'series' and in 'parallel', including standby redundancy where necessary. Consider the following system.



Here the system consists of four components S_1,S_2,S_3,S_4 and it functions if S_1 and S_2 operate or S_3 and S_4 operate. If the individual reliabilities are 0.9 and the switches all operate independently, what is the reliability of the system?

The system fails if **both** the upper part (S_1, S_2) and the lower part (S_3, S_4) fail. We have already seen, in Example 2.4, that the reliability of the upper part is given by

$$\begin{aligned} P(\mathrm{S}_1 \text{ and } \mathrm{S}_2 \text{ operate}) &= P(\mathrm{S}_1 \text{ operates}) \times P(\mathrm{S}_2 \text{ operates}) \\ &= 0.9 \times 0.9 \\ &= 0.81 \end{aligned}$$

so that the probability that the upper part fails is 0.19. Similarly, the probability that the lower part fails is also 0.19. The probability that the system fails is now given by

$$\begin{aligned} P(\text{system fails}) &= P(\text{upper part fails and lower part fails}) \\ &= P(\text{upper part fails}) \times P(\text{lower part fails}) \\ &= 0.19 \times 0.19 \\ &= 0.0361 \end{aligned}$$

so its reliability is $1 - 0.0361 = 0.9639$ or 96.4%.

In general, if the probabilities of working for S_1,S_2,S_3,S_4 are p_1, p_2, p_3, p_4 respectively, the reliability of such a system is given by

$$1 - (1 - p_1p_2)(1 - p_3p_4)$$

and, if $p_1 = p_2 = p_3 = p_4 = p$, the reliability is $1 - (1 - p^2)^2$.

♡ **Example** 2.7. An engineer has designed a storm water sewer system so that the yearly maximum discharge will cause flooding on average once every 10 years. This means that the probability each year that there will be a discharge which causes flooding is 0.1. If it can be assumed that the maximum discharges are independent from year to year, what is the probability that there will be at least one flood in the next five years.

Whenever we require "the probability of **at least** one", it is simpler to determine "the probability of none" and then subtract this from 1. In this case, the probability of no flood in any particular year is $1 - 0.1 = 0.9$, so that the probability of no flood in 5 years is

$$\begin{aligned} P(\text{No flood in 5 years}) &= P(\text{No flood in year 1 } \textbf{and} \text{ no flood in year 2 } \textbf{and} \cdots \\ &\qquad \cdots \textbf{ and} \text{ no flood in year 5}) \\ &= P(\text{No flood in year 1}) \times P(\text{No flood in year 2}) \times \cdots \\ &\qquad \cdots \times P(\text{No flood in year 5}) \\ &= 0.9 \times 0.9 \times 0.9 \times 0.9 \times 0.9 = 0.9^5 = 0.59 \end{aligned}$$

and therefore

$$P(\text{At least one flood in 5 years}) = 1 - 0.59 = 0.41$$

Although the sewer system has been designed to withstand a flood which occurs on average once every 10 years, the probability that this will occur within the next 5 years is just over 0.4.

The ideas of **design life**, **reliability** and **return period** will be covered in more detail in a later chapter.

2.2 Conditional Probability and Bayes Theorem

2.2.1 Conditional Probability

The probability of an event B occurring when it is known that some event A has already occurred is called a **conditional probability** and it is denoted by $P(B|A)$. The symbol $P(B|A)$ is usually read as "the probability that B occurs given that A has already occurred', or simply, the probability of B given A.

The formula for finding the conditional probability is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided } P(A) > 0. \tag{2.1}$$

♡ **Example** 2.8. The probability that a plane departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it arrives and departs on time is $P(D \cap A) = 0.78$.

The probability that a plane departed on time given that it arrived on time is:

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95.$$

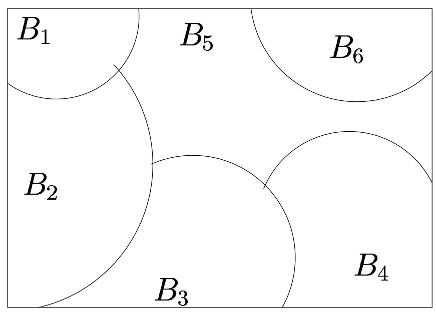
The probability that a plane arrives on time given that it departed on time is:

$$P(A|D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94.$$

2.2.2 Theorem of Total Probability

Two events B_1 and B_2 are called **mutually exclusive** if they cannot occur simultaneously. For example, let A denote the event that head turns up and B denote the event that tail turns up when a coin is tossed. Here $P(B_1 \cap B_2) = 0$.

Sometimes we partition (i.e. divide) the sample space by mutually exclusive events. Often a set of such events covering the entire sample space, called **a set of exhaustive** events, are considered. For example, suppose that $B_1, \ldots, B_k$ denote a set of mutually exclusive and exhaustive events. So $B_1 \cup B_2 \cup \cdots \cup B_k = S$ where S is the sample space. In the coin tossing example, B_1 and B_2 provide a set of mutually exclusive and exhaustive events.



To find the probability of another event A (other than the $B_1, \ldots, B_k$), intuition suggests that we can find the intersection probability of A with each of $B_1, \ldots, B_k$ and add them up. The **theorem of total probability** is exactly that and is as follows:

If the events $B_1, \ldots, B_k$ form a partition of the sample space such that $P(B_i) \neq= 0, i = 1, \ldots, k$, then for any event A in the sample space S:

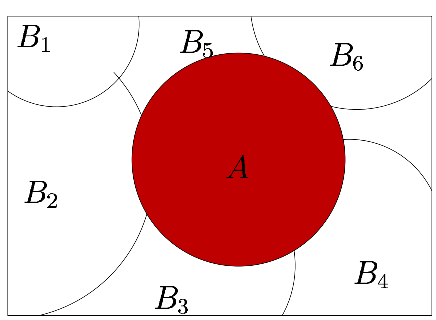
$$P(A) = \sum_{i=1}^{k} P(B_i \cap A).$$

However, using the definition of conditional probability in (2.1) we have:

$$P(B_i \cap A) = P(B_i)P(A|B_i).$$

Hence we have:

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$



♡ **Example** 2.9. In a certain assembly plant, three machines B_1, B_2, and B_3 make 30%, 45%, and 25%, respectively of the products. It is known from past experience that 2%, 3% and 2% of the products made by each machine, respectively, are defective. Now suppose that a finished product is randomly selected. What is the probability that it is defective?

Consider the following events:

A: the product is defective,

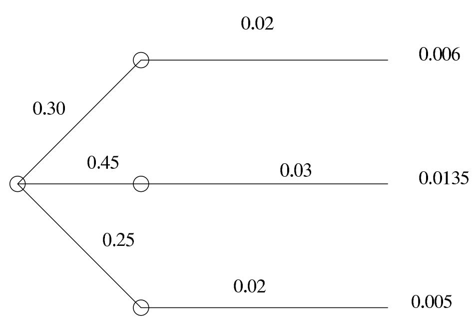
B_1: the product is made by machine B_1,

B_2: the product is made by machine B_2,

B_3: the product is made by machine B_3,

Using the theorem of total probability:

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$



But we have:

$$\begin{aligned} P(B_1) &= 0.30, \quad P(A|B_1) = 0.02 \\ P(B_2) &= 0.45, \quad P(A|B_1) = 0.03 \\ P(B_3) &= 0.25, \quad P(A|B_3) = 0.02 \end{aligned}$$

Hence

$$\begin{aligned} P(B_1)\,P(A|B_1) &= (0.30)(0.02) = 0.006 \\ P(B_2)\,P(A|B_2) &= (0.45)(0.03) = 0.0135 \\ P(B_3)\,P(A|B_3) &= (0.25)(0.02) = 0.005. \end{aligned}$$

and hence:

$$P(A) = 0.006 + 0.00135 + 0.0005 = 0.0245.$$

If instead, we wanted to find the inverse probability that $P(B_1|A)$, i.e. the probability that a randomly selected product was made by machine B_1 given that it is defective? We apply the Bayes the theorem to find the inverse probability.

2.2.3 Bayes Theorem

Let $B_1, B_2, \ldots, B_k$ be a set of mutually exclusive and exhaustive events. For any new event A,

$$P(B_r|A) = \frac{P(B_r \cap A)}{P(A)} = \frac{P(A|B_r)P(B_r)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}, \quad r = 1, \ldots, k. \tag{2.2}$$

♡ **Example** 2.10. For the above example with three machines:

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(A)} = \frac{(0.30)(0.02)}{0.0245} = 0.2449.$$

So, although there was a 30% chance that a randomly selected product was made by machine B_1, the probability that a randomly selected product was made by machine B_1 given that the product was defective reduces to 24.49%. This is to be expected since machine B_1 produces less defective products than some others.

If, instead, we suppose that machine B_1 produces 5% defective items. Then

$$P(A) = (0.30)(0.05) + 0.00135 + 0.0005 = 0.01685, \text{ and}$$
$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(A)} = \frac{(0.30)(0.05)}{0.01685} = 0.471.$$

Here the probability that a randomly selected product was made by machine B_1 given that the product was defective increases to 47.10%.

$P(B_1)$ and $P(B_1|A)$ are called the **prior** and **posterior** probability, respectively.

♡ **Example** 2.11. Consider a disease that is thought to occur in 1% of the population. Using a particular blood test a physician observes that out of the patients with disease 98% possess a particular symptom. Also assume that 0.1% of the population without the disease have the same symptom. A randomly chosen person from the population is blood tested and is shown to have the symptom. What is the conditional probability that the person has the disease?

Let B_1 be the event that a randomly chosen person has the disease and B_2 is the complement of B_1. Let A be the event that a randomly chosen person has the symptom. The problem is to determine $P(B_1|A)$.

We have $P(B_1) = 0.01$ since 1% of the population has the disease, and $P(A|B_1) = 0.98$. Also $P(B_2) = 0.99$ and $P(A|B_2) = 0.001$. Now

$$\begin{aligned} P(\text{disease} \mid \text{symptom}) = P(B_1|A) &= \frac{P(A|B_1)\,P(B_1)}{P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2)} \\ &= \frac{0.98 \times 0.01}{0.98 \times 0.01 + 0.001 \times 0.99} \\ &= 0.9082. \end{aligned}$$

So the unconditional probability of disease, $P(B_1) = 0.01 = 1\%$, has increased to 90.82% when the symptom is present, $P(B_1|A)$.