Revision of Lecture Nine

- AWGN channel: decision variable $r_k = \bar{r}_k + n_k = g_0 s_k + n_k$, where channel g_0 is a known constant, n_k AWGN with PSD $\frac{N_0}{2}$, average signal power or energy per symbol is E_s, and (average) channel SNR $\tilde{\Lambda} = \frac{E_s}{N_0}$

- Bit error ratio performance:

BPSK $P_e = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = Q\left(\sqrt{2\tilde{\Lambda}}\right)$

QPSK $P_e = Q\left(\sqrt{\frac{E_s}{N_0}}\right) = Q\left(\sqrt{\tilde{\Lambda}}\right)$

16QAM $P_e = \frac{3}{4}Q\left(\sqrt{\frac{E_s}{5N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{9E_s}{5N_0}}\right) - \frac{1}{4}Q\left(\sqrt{\frac{25E_s}{5N_0}}\right) \approx \frac{3}{4}Q\left(\sqrt{\frac{\tilde{\Lambda}}{5}}\right)$

- This lecture we consider fading performance

Performance in Flat Rayleigh Fading Channels

- A narrowband channel is represented by $c(t) = \alpha(t) \cdot e^{j\phi(t)}$. Assume that fading is sufficiently slow, $c(t)$ and $\phi(t)$ are symbol invariant $\rightarrow$ during one symbol period

$$c = \alpha \cdot e^{j\phi}$$

- The Rayleigh fading envelope α has a PDF

$$p_\alpha(\alpha) = \frac{\alpha}{\alpha_0^2} \exp\left(-\frac{\alpha^2}{2\alpha_0^2}\right), \quad \alpha \geq 0$$

and the channel phase ϕ is uniformly distributed in $[-\pi,\ \pi]$ with PDF

$$p_\phi(\phi) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \phi \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

- Note that Re$[c]$ and Im$[c]$ are i.i.d. Gaussian with variance α_0^2, and α has mean $E[\alpha] = \bar{\alpha} = \sqrt{\frac{\pi}{2}}\alpha_0$, 2nd moment $E[\alpha^2] = 2\alpha_0^2$ and variance $\frac{4-\pi}{2}\alpha_0^2$

Flat Fading Performance (continue)

- Given the transmitted baseband signal $m(t)$ with average energy E_s, the received signal is

$$r(t) = \alpha \cdot e^{j\phi} \cdot m(t) + n(t)$$

where $n(t)$ is AWGN with PSD $\frac{N_0}{2}$

- Define the instantaneous channel SNR

$$\lambda = \alpha^2 \frac{E_s}{N_0}$$

Note that $\lambda > 0$ is a chi-square distribution with PDF

$$p_\lambda(\lambda) = \frac{1}{\Lambda} e^{-\lambda/\Lambda}$$

- The **average channel SNR** Λ is defined as

$$\Lambda = E[\lambda] = \bar{\lambda} = 2\alpha_0^2 \frac{E_s}{N_0}$$

- Let $P(\lambda)$ be the instantaneous error probability, the **average error probability** is then defined as

$$P_e = \int_0^\infty P(\lambda) p_\lambda(\lambda) d\lambda = \frac{1}{\Lambda} \int_0^\infty P(\lambda) e^{-\lambda/\Lambda} d\lambda$$

4QAM Flat Fading Performance

- **Fading does not change the decision boundaries** $I, Q = 0$, but $I, Q = d$ becomes $I, Q = \alpha d$
- Using non-fading result $P_e = Q(d/\sqrt{N_0/2}) \rightarrow$ the **instantaneous error probability**

$$P_e(\lambda) = Q\left(\frac{\alpha d}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\lambda}\right)$$

- Note $\lambda \geq 0$ and $\Lambda = E[\lambda] = 2\alpha_0^2 E_s/N_0$, the **average error probability** is

$$P_e = \frac{1}{\Lambda}\int_0^{\infty} Q\left(\sqrt{\lambda}\right) e^{-\lambda/\Lambda} d\lambda$$

- Note that $\lambda = \alpha^2 E_s/N_0$ and $E_s = 2d^2$, the close-form for P_e is:

$$P_e = \int_0^{\infty} Q\left(\frac{\alpha d}{\sqrt{N_0/2}}\right) p_\alpha(\alpha) d\alpha = \int_0^{\infty} Q\left(\frac{\alpha d}{\sqrt{N_0/2}}\right) \frac{\alpha}{\alpha_0^2} \exp\left(-\frac{\alpha^2}{2\alpha_0^2}\right) d\alpha$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{2\alpha_0^2 \frac{d^2}{N_0}}{1 + 2\alpha_0^2 \frac{d^2}{N_0}}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{\Lambda}{2 + \Lambda}}\right)$$

4QAM Flat Fading Performance (derivation)

- Note

integration formula $$\int_0^\infty 2Q\left(\sqrt{2}\beta x\right) e^{-\mu x^2} x\, dx = \frac{1}{2\mu}\left(1 - \frac{\beta}{\sqrt{\mu+\beta^2}}\right)$$

average error probability $$P_e = \int_0^\infty Q\left(\frac{\alpha d}{\sqrt{N_0/2}}\right)\frac{\alpha}{\alpha_0^2}\exp\left(-\frac{\alpha^2}{2\alpha_0^2}\right) d\alpha$$

$$= \int_0^\infty 2Q\left(\sqrt{2}\frac{d\alpha_0}{\sqrt{N_0/2}}\frac{\alpha}{\sqrt{2}\alpha_0}\right)\exp\left(-\frac{\alpha^2}{2\alpha_0^2}\right)\frac{\alpha}{\sqrt{2}\alpha_0} d\left(\frac{\alpha}{\sqrt{2}\alpha_0}\right)$$

- Letting $\mu = 1$, $\beta = \frac{d\,\alpha_0}{\sqrt{N_0/2}}$ and $x = \frac{\alpha}{\sqrt{2}\alpha_0}$ in the above **integration formula** leads to

$$P_e = \frac{1}{2}\left(1 - \frac{\frac{d\,\alpha_0}{\sqrt{N_0/2}}}{\sqrt{1+\left(\frac{d\,\alpha_0}{\sqrt{N_0/2}}\right)^2}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{2\alpha_0^2\frac{d^2}{N_0}}{1+2\alpha_0^2\frac{d^2}{N_0}}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{\Lambda}{2+\Lambda}}\right)$$

where $\Lambda = 2\alpha_0^2\frac{E_s}{N_0}$, $E_s = 2d^2 \Rightarrow \Lambda = \frac{4\alpha_0^2 d^2}{N_0}$

4QAM Fading / Non-Fading BER Comparison

- In fading, average SNR $\Lambda = 2\alpha_0^2 \frac{E_s}{N_0}$

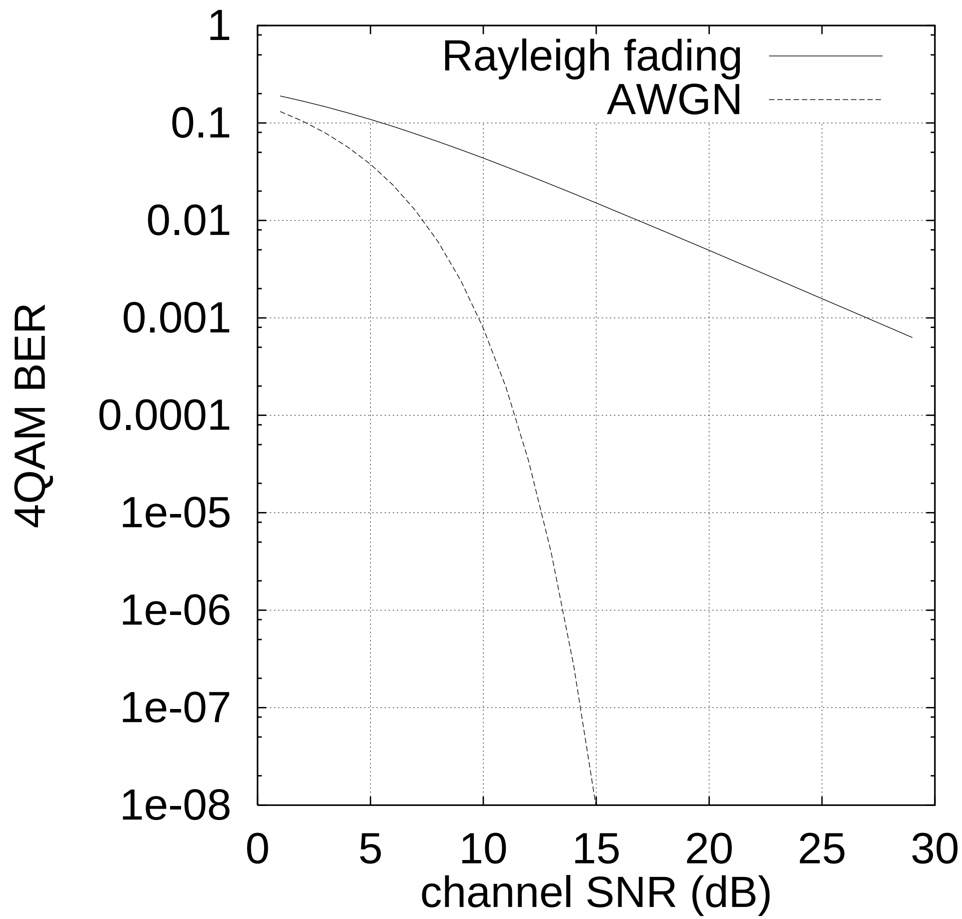
  Average BER

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\Lambda}{2+\Lambda}}\right)$$

- Compare with AWGN, average SNR is defined as $\tilde{\Lambda} = E_s/N_0$

  Average error probability

$$P_e = Q\left(\sqrt{\tilde{\Lambda}}\right)$$



It is clear that fading is “a big **killer**” limiting communication system’s performance

16QAM Flat Fading Performance: C1 BER

- For C1 bits, the decision boundaries are not changed:

$$I,Q > 0 \rightarrow i_1, q_1 = 0, \;\; I,Q \le 0 \rightarrow i_1, q_1 = 1$$

But $\pm d$, $\pm 3d$ become $\pm \alpha d$, $\pm 3\alpha d$. Also $E_s = 10d^2$ and $\Lambda = 20\alpha_0^2 d^2/N_0$

- Thus, the instantaneous C1 error probability is

$$P_{e,1}(\lambda) = \frac{1}{2}Q\left(\frac{\alpha d}{\sigma_n}\right) + \frac{1}{2}Q\left(\frac{3\alpha d}{\sigma_n}\right) = \frac{1}{2}Q\left(\sqrt{\frac{\lambda}{5}}\right) + \frac{1}{2}Q\left(3\sqrt{\frac{\lambda}{5}}\right)$$

- The average C1 error probability is therefore

$$P_{e,1} = \frac{1}{\Lambda}\int_0^{\infty} P_{e,1}(\lambda) e^{-\lambda/\Lambda} d\lambda = \frac{1}{4}\left(1 - \sqrt{\frac{2\alpha_0^2 \frac{d^2}{N_0}}{1 + 2\alpha_0^2 \frac{d^2}{N_0}}}\right) + \frac{1}{4}\left(1 - \sqrt{\frac{18\alpha_0^2 \frac{d^2}{N_0}}{1 + 18\alpha_0^2 \frac{d^2}{N_0}}}\right)$$

$$= \frac{1}{4}\left(1 - \sqrt{\frac{\Lambda}{10 + \Lambda}}\right) + \frac{1}{4}\left(1 - \sqrt{\frac{9\Lambda}{10 + 9\Lambda}}\right)$$

16QAM Fading Performance: C2 BER

- For C2 bits, the decision boundaries are changed to:

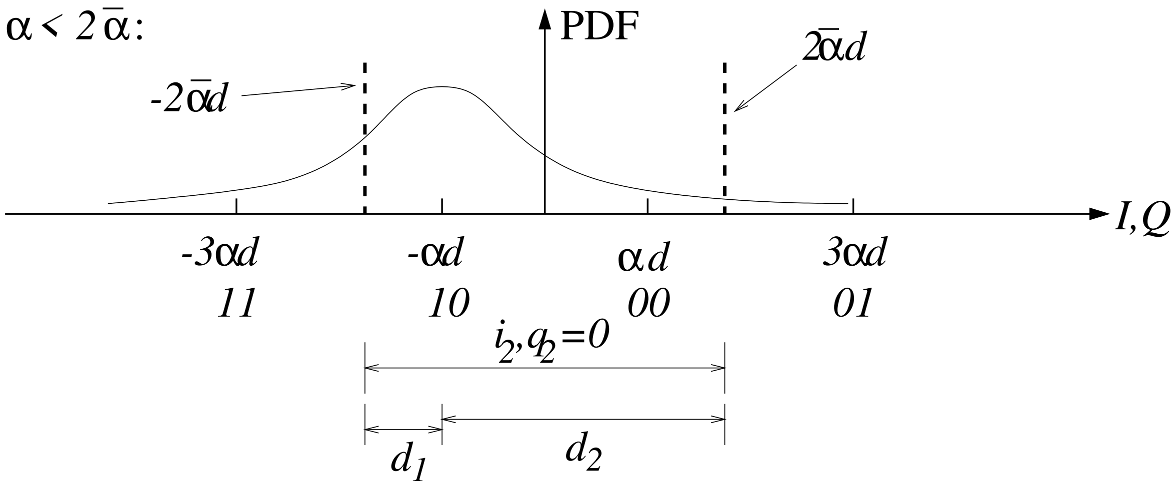
$$I, Q > 2\bar{\alpha}d \text{ or } I, Q \leq -2\bar{\alpha}d \rightarrow i_2, q_2 = 1$$

$$-2\bar{\alpha}d < I, Q \leq 2\bar{\alpha}d \rightarrow i_2, q_2 = 0$$

Note that the average value of α, i.e. $\bar{\alpha}$, has to be used for decision threshold

- Two cases of $i_2, q_2 = 0$ error need consideration

1. $i_2, q_2 = 0$ error in case of $\alpha < 2\bar{\alpha}$
   - Instantaneous symbols $-\alpha d$ and αd are within region defined by two decision boundaries
   - Error occurs when noise makes received signal outside the region and instantaneous C2 bit $= 0$ error probability for $\alpha < 2\bar{\alpha}$ is



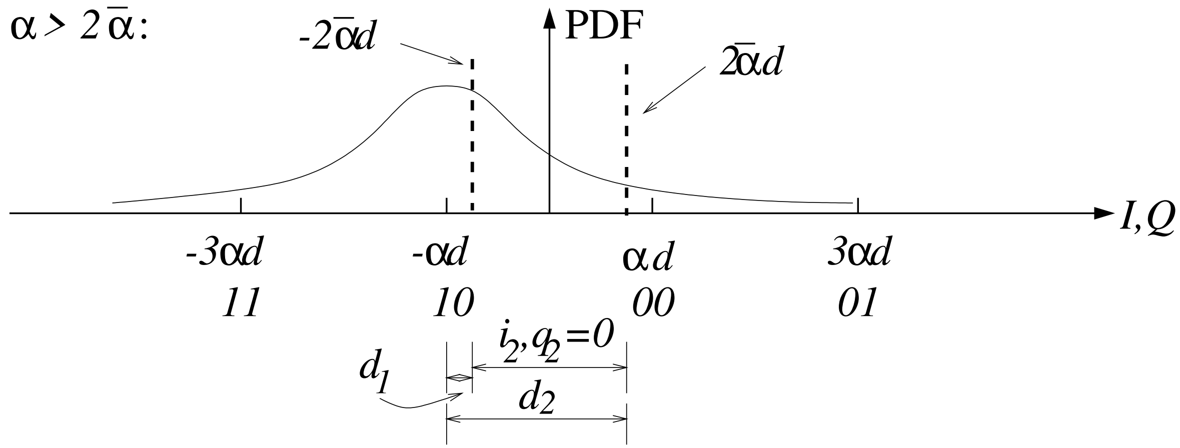
$$P_{2,0,<2}(\alpha) = Q\left(d_1/\sqrt{N_0/2}\right) + Q\left(d_2/\sqrt{N_0/2}\right) = Q\left(\frac{(2\bar{\alpha} - \alpha)d}{\sqrt{N_0/2}}\right) + Q\left(\frac{(2\bar{\alpha} + \alpha)d}{\sqrt{N_0/2}}\right)$$

16QAM: C2 BER (continue)

2. $i_2, q_2 = 0$ error in the case of $\alpha > 2\bar{\alpha}$: Instantaneous symbols $-\alpha d$ and αd are outside region defined by two decision boundaries

   – Correct decision occurs only when noise moves received signal inside the region

   – Thus instantaneous C2 bit $= 0$ error probability for $\alpha > 2\bar{\alpha}$ is



$$P_{2,0,>2}(\alpha) =$$

$$1 - Q\left(d_1/\sqrt{N_0/2}\right) + Q\left(d_2/\sqrt{N_0/2}\right) = 1 - Q\left(\frac{(-2\bar{\alpha}+\alpha)d}{\sqrt{N_0/2}}\right) + Q\left(\frac{(2\bar{\alpha}+\alpha)d}{\sqrt{N_0/2}}\right)$$

- Note $\alpha > 0$, the average error for $i_2, q_2 = 0$ is therefore

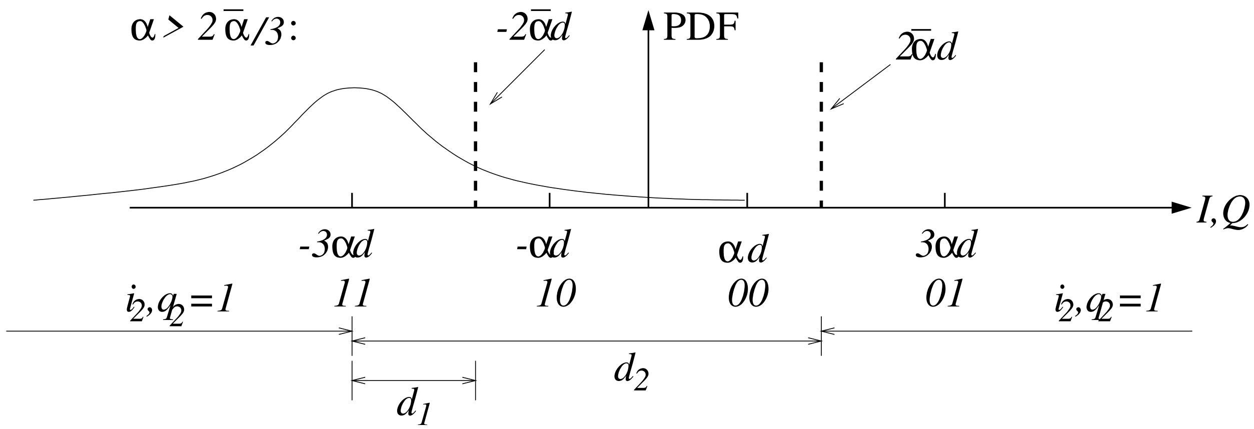
$$P_{2,0} = \int_0^{2\bar{\alpha}} P_{2,0,<2}(\alpha)p_\alpha(\alpha)d\alpha + \int_{2\bar{\alpha}}^{\infty} P_{2,0,>2}(\alpha)p_\alpha(\alpha)d\alpha$$

There exists closed form solution for this integration but it is very complicated

16QAM: C2 BER (continue)

- Two cases of $i_2, q_2 = 1$ error need consideration

1. $i_2, q_2 = 1$ error in case of $\alpha > 2\bar{\alpha}/3$
   - Instantaneous symbols $-3\alpha d$ and $3\alpha d$ are within correct regions corresponding to respective decision boundaries
   - Error occurs when noise makes received signal outside the corresponding region



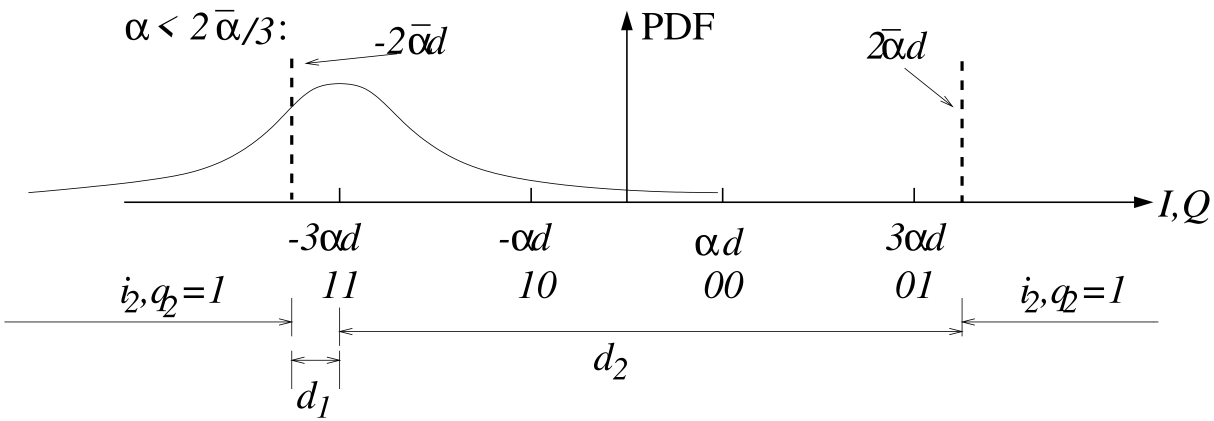
   - Thus instantaneous C2 bit $= 1$ error probability for $\alpha > 2\bar{\alpha}/3$ is

$$P_{2,1,>2/3}(\alpha) = Q\left(d_1/\sqrt{N_0/2}\right) - Q\left(d_2/\sqrt{N_0/2}\right)$$

$$= Q\left(\frac{(3\alpha - 2\bar{\alpha})d}{\sqrt{N_0/2}}\right) - Q\left(\frac{(3\alpha + 2\bar{\alpha})d}{\sqrt{N_0/2}}\right)$$

16QAM: C2 BER (continue)

2. $i_2, q_2 = 1$ error in case of $\alpha < 2\bar{\alpha}/3$
   - Instantaneous symbols $-3\alpha d$ and $3\alpha d$ are outside correct regions defined by respective decision boundaries
   - Correct decision occurs only when noise moves received signal to the correct region



   - The instantaneous C2 bit $= 1$ error probability for $\alpha < 2\bar{\alpha}/3$ is

$$P_{2,1,<2/3}(\alpha) = 1 - Q\left(d_1/\sqrt{N_0/2}\right) - Q\left(d_2/\sqrt{N_0/2}\right)$$

$$= 1 - Q\left(\frac{(2\bar{\alpha} - 3\alpha)d}{\sqrt{N_0/2}}\right) - Q\left(\frac{(2\bar{\alpha} + 3\alpha)d}{\sqrt{N_0/2}}\right)$$

- The average error for $i_2, q_2 = 1$ is therefore

$$P_{2,1} = \int_0^{2\bar{\alpha}/3} P_{2,1,<2/3}(\alpha)p_\alpha(\alpha)d\alpha + \int_{2\bar{\alpha}/3}^{\infty} P_{2,1,>2/3}(\alpha)p_\alpha(\alpha)d\alpha$$

16QAM Fading BER (continue)

- Thus average C2 error probability is

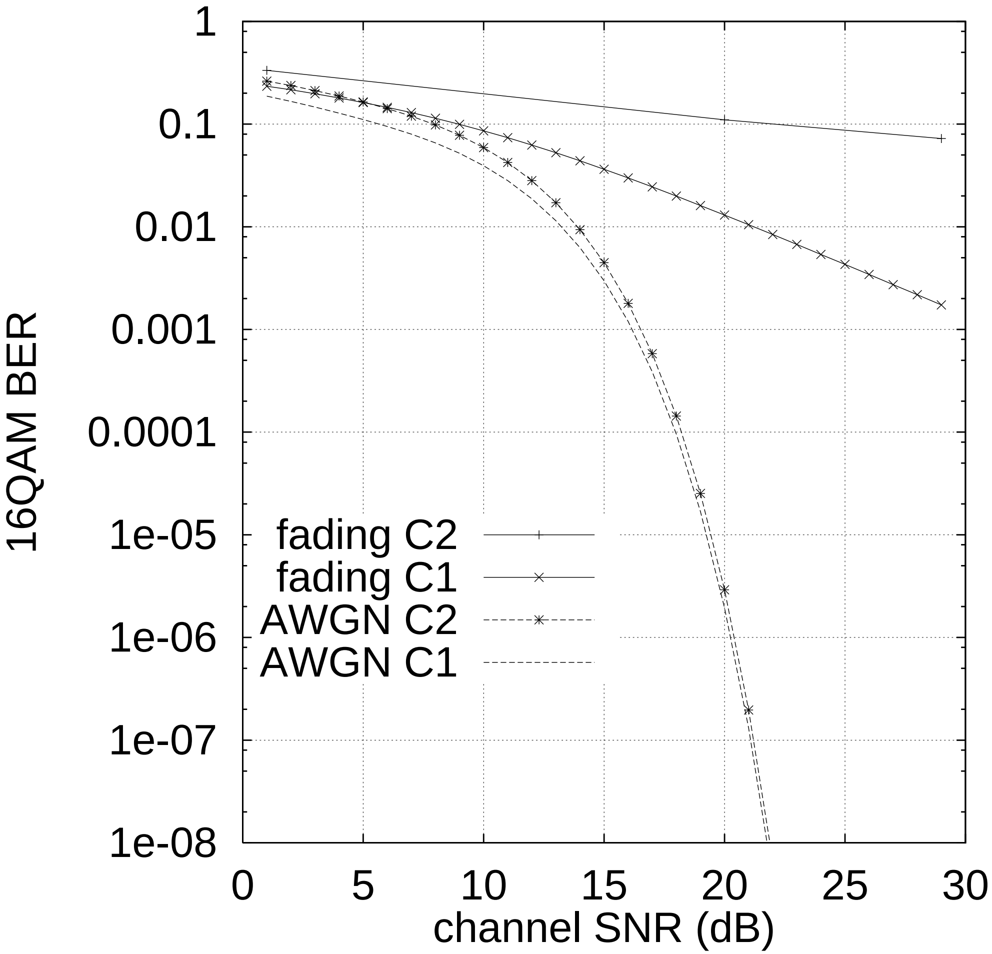
$$P_{e,2} = \frac{1}{2}(P_{2,0} + P_{2,1})$$

- Therefore the average error probability for 16QAM is

$$P_e = \frac{1}{2}(P_{e,1} + P_{e,2})$$

- 16QAM fading / non-fading BER comparison:

- Fading degrades BER performance seriously $\Rightarrow$ counter fading measures



Alternatively, Monte Carlo simulation is often used to evaluate fading BER

Recall slide 35 for flat Rayleigh fading channel simulation

Summary

- Narrowband Rayleigh fading channel: fading envelope and phase PDFs, instantaneous and average channel SNRs, instantaneous and average error probabilities

- 4QAM fading performance: decision boundaries remain $I, Q = 0$, instantaneous error probability by substituting αd for d in AWGN result, close-form average error probability

- 16QAM fading performance:

  ⋆ C1 decision boundaries remain $I, Q = 0$, C1 instantaneous error probability by substituting αd and $3\alpha d$ for d and $3d$ in AWGN result, C1 close-form average error probability

  ⋆ C2 decision boundaries are changed from $I, Q = \pm 2d$ to $I, Q = \pm 2\bar{\alpha} d$

  Instantaneous error probability for $i_2, q_2 = 0$: (1) $\alpha < 2\bar{\alpha}$, (2) $\alpha > 2\bar{\alpha}$; average error probability

  Instantaneous error probability for $i_2, q_2 = 1$: (1) $\alpha < 2\bar{\alpha}/3$, (2) $\alpha > 2\bar{\alpha}/3$; average error probability