Minimum Mean Square Error Equalisation

The discrete-time channel model is given by

$$r(k) = \sum_{i=0}^{n_c} c_i s(k-i) + n(k) \tag{1}$$

where n_c is the channel length, c_i are complex-valued channel taps, the N-QAM symbol $s(k) \in \{s_{i,l} = u_i + ju_l,\ 1 \le i, l \le \sqrt{N}\}$ with $j = \sqrt{-1}$, $u_i = 2i - \sqrt{N} - 1$ and $u_l = 2l - \sqrt{N} - 1$, while $n(k)$ is the complex-valued AWGN with $E[|n(k)|^2] = 2\sigma_n^2$.

The linear equaliser is given by

$$y(k) = \sum_{i=0}^{M} w_i^* r(k-i) = \mathbf{w}^H \mathbf{r}(k) = \mathbf{r}^T(k)\mathbf{w}^*, \tag{2}$$

where M is the equaliser order, w_i are complex-valued equaliser weights, $\mathbf{w} = [w_0\ w_1 \cdots w_M]^T$ and $\mathbf{r}(k) = [r(k)\ r(k-1) \cdots r(k-M)]^T$. The equaliser output $y(k)$ is passed to the decision device to produce an estimate $\hat{s}(k-\tau)$ of the transmitted symbol $s(k-\tau)$, where $0 \le \tau \le L$ is the equaliser's decision delay with $L = n_c + M$.

The received signal vector $\mathbf{r}(k)$ can be expressed by the well-known signal model

$$\mathbf{r}(k) = \mathbf{C}\,\mathbf{s}(k) + \mathbf{n}(k) \tag{3}$$

where the noise vector is $\mathbf{n}(k) = [n(k)\ n(k-1) \cdots n(k-M)]^T$, the tranmitted symbol vector is $\mathbf{s}(k) = [s(k)\ s(k-1) \cdots s(k-L)]^T$, and the $(M+1) \times (L+1)$ CIR convolution matrix has the Toeplitz form

$$\mathbf{C} = \begin{bmatrix} c_0 & c_1 & \cdots & c_M & 0 & \cdots & 0 \\ 0 & c_0 & c_1 & \cdots & c_M & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_0 & c_1 & \cdots & c_M \end{bmatrix} = [\mathbf{c}_0\ \mathbf{c}_1 \cdots \mathbf{c}_L]\,. \tag{4}$$

The MSE $J_{\mathrm{MSE}}(\mathbf{w}) = E[|s(k-\tau) - y(k)|^2]$ is expressed as

$$\begin{aligned} J_{\mathrm{MSE}}(\mathbf{w}) &= E[(s(k-\tau) - y(k))(s^*(k-\tau) - y^*(k))] \\ &= E[s(k-\tau)s^*(k-\tau)] - E[y(k)s^*(k-\tau)] - E[y^*(k)s(k-\tau)] + E[y(k)y^*(k)]. \end{aligned} \tag{5}$$

The first term is the symbol energy $E[s(k-\tau)s^*(k-\tau)] = \sigma_s^2$, the second term can be expressed as

$$E[s^*(k-\tau)\mathbf{w}^H(\mathbf{C}\,\mathbf{s}(k) + \mathbf{n}(k))] = \sigma_s^2 \mathbf{w}^H \mathbf{c}_\tau, \tag{6}$$

and similarly the third term is

$$E[s(k-\tau)\mathbf{w}^T(\mathbf{C}^*\mathbf{s}^*(k) + \mathbf{n}^*(k))] = \sigma_s^2 \mathbf{w}^T \mathbf{c}_\tau^*, \tag{7}$$

while the last term can be expressed as

$$E[\mathbf{w}^H(\mathbf{C}\,\mathbf{s}(k) + \mathbf{n}(k))(\mathbf{s}^H(k)\mathbf{C}^H + \mathbf{n}^H(k))\mathbf{w}] = \sigma_s^2 \mathbf{w}^H \mathbf{C}\mathbf{C}^H \mathbf{w} + 2\sigma_n^2 \mathbf{w}^H \mathbf{I}_{M+1} \mathbf{w} \tag{8}$$

with $\mathbf{I}_{M+1}$ being the $(M+1) \times (M+1)$ identity matrix. Thus, the MSE criterion is given by

$$J_{\mathrm{MSE}}(\mathbf{w}) = \sigma_s^2 \left(1 - \mathbf{w}^H \mathbf{c}_\tau - \mathbf{w}^T \mathbf{c}_\tau^* + \mathbf{w}^H \left(\mathbf{C}\mathbf{C}^H + \frac{2\sigma_n^2}{\sigma_s^2} \mathbf{I}_{M+1} \right) \mathbf{w} \right). \tag{9}$$

The optimal MMSE solution minimises the MSE criterion

$$\mathbf{w}_{\mathrm{MMSE}} = \arg \min_{\mathbf{w}} J_{\mathrm{MSE}}(\mathbf{w}). \tag{10}$$

The MMSE solution is obtained by solving the vector equation

$$\bigtriangledown J_{\mathrm{MSE}}(\mathbf{w})|_{\mathbf{w}=\mathbf{w}_{\mathrm{MMSE}}} = \mathbf{0}, \tag{11}$$

where the gradient vector

$$\bigtriangledown J_{\mathrm{MSE}}(\mathbf{w}) = \left[\frac{\partial J_{\mathrm{MSE}}}{\partial w_0} \ \frac{\partial J_{\mathrm{MSE}}}{\partial w_1} \cdots \frac{\partial J_{\mathrm{MSE}}}{\partial w_M} \right]^T. \tag{12}$$

Note that $w_i = w_{R,i} + j w_{I,i}$ and

$$\frac{\partial J_{\mathrm{MSE}}}{\partial w_i} = \frac{1}{2} \left(\frac{\partial J_{\mathrm{MSE}}}{\partial w_{R,i}} + j \frac{\partial J_{\mathrm{MSE}}}{\partial w_{I,i}} \right). \tag{13}$$

Further denote the $(M+1) \times (M+1)$ matrix $\mathbf{D} = [d_{q,l}]$ as

$$\mathbf{D} = \mathbf{C}\mathbf{C}^H + \frac{2\sigma_n^2}{\sigma_s^2} \mathbf{I}_{M+1}, \tag{14}$$

and the $(\tau+1)$th column of $\mathbf{C}$ as $\mathbf{c}_\tau = [\tilde{c}_{0,\tau} \ \tilde{c}_{1,\tau} \cdots \tilde{c}_{M,\tau}]^T$. Then the MSE can be expressed as

$$J_{\mathrm{MSE}}(\mathbf{w}) = \sigma_s^2 \left(1 - \sum_{l=0}^{M} w_l^* \tilde{c}_{l,\tau} - \sum_{l=0}^{M} w_l \tilde{c}_{l,\tau}^* + \sum_{q=0}^{M} \sum_{l=0}^{M} w_q^* w_l d_{q,l} \right). \tag{15}$$

Noting

$$\frac{\partial J_{\mathrm{MSE}}}{\partial w_{R,i}} = -\tilde{c}_{i,\tau} \frac{\partial w_i^*}{\partial w_{R,i}} - \tilde{c}_{i,\tau}^* \frac{\partial w_i}{\partial w_{R,i}} + \frac{\partial w_i^*}{\partial w_{R,i}} \sum_{l=0}^{M} w_l d_{i,l} + \frac{\partial w_i}{\partial w_{R,i}} \sum_{q=0}^{M} w_q^* d_{q,i}, \tag{16}$$

$$\frac{\partial J_{\mathrm{MSE}}}{\partial w_{I,i}} = -\tilde{c}_{i,\tau} \frac{\partial w_i^*}{\partial w_{I,i}} - \tilde{c}_{i,\tau}^* \frac{\partial w_i}{\partial w_{I,i}} + \frac{\partial w_i^*}{\partial w_{I,i}} \sum_{l=0}^{M} w_l d_{i,l} + \frac{\partial w_i}{\partial w_{I,i}} \sum_{q=0}^{M} w_q^* d_{q,i}, \tag{17}$$

as well as

$$\frac{\partial w_i^*}{\partial w_{R,i}} = \frac{\partial w_i}{\partial w_{R,i}} = 1, \quad \frac{\partial w_i^*}{\partial w_{I,i}} = -j, \quad \frac{\partial w_i}{\partial w_{I,i}} = j, \tag{18}$$

we have

$$\begin{aligned} \frac{\partial J_{\mathrm{MSE}}}{\partial w_i} &= \frac{1}{2} \left(-\tilde{c}_{i,\tau}(1 - j \cdot j) - \tilde{c}_{i,\tau}^*(1 + j \cdot j) + (1 - j \cdot j) \sum_{l=0}^{M} w_l d_{i,l} + (1 + j \cdot j) \sum_{q=0}^{M} w_q^* d_{q,i} \right) \\ &= -\tilde{c}_{i,\tau} + \sum_{l=0}^{M} w_l d_{i,l}. \end{aligned} \tag{19}$$

Thus

$$\bigtriangledown J_{\mathrm{MSE}}(\mathbf{w}) = -\mathbf{c}_\tau + \mathbf{D}\mathbf{w}. \tag{20}$$

From $-\mathbf{c}_\tau + \mathbf{D}\mathbf{w}_{\mathrm{MMSE}} = \mathbf{0}$, we obtain the MMSE solution $\mathbf{w}_{\mathrm{MMSE}} = \mathbf{D}^{-1}\mathbf{c}_\tau$, or

$$\mathbf{w}_{\mathrm{MMSE}} = \left(\mathbf{C}\mathbf{C}^H + \frac{2\sigma_n^2}{\sigma_s^2} \mathbf{I}_{M+1} \right)^{-1} \mathbf{c}_\tau. \tag{21}$$