Robust $\mathcal{H}_\infty$ Control for Networked Control Systems with Uncertainties and Multiple-Packet Transmission

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Abstract

A class of networked control systems is investigated where the plant has time-varying norm-bounded parameter uncertainties, both the sensor-to-controller and controller-to-actuator channels implement multiple-packet transmission and experience random packet dropouts. Sufficient conditions for synthesis of robust stochastic stabilisation and design of robust $\mathcal{H}_\infty$ controller are derived in the form of linear matrix inequalities. An example is provided to demonstrate the effectiveness of our proposed method.

Index Terms

Networked control systems, norm-bounded uncertainties, packet dropouts, multiple-packet transmission, robust $\mathcal{H}_\infty$ control

I. INTRODUCTION

A networked control system (NCS) [1]–[5] is a control system in which the control loop is closed via a shared communication network. Compared to the conventional point-to-point system connection, the use of an NCS has advantages of low installation cost, reducing system wiring, simple system diagnosis and easy maintenance. However, some inherent shortcomings of NCSs, such as bandwidth constraints, packet dropouts and packet delays, will degrade performance of NCSs or even cause instability. Packet dropouts, which can randomly occur due to node failures or network congestion, impose one of the most important issues in NCSs. Stochastic approaches

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based on the mean square stability [6], [7] are typically adopted to deal with packet dropouts. Under a stochastic approach, the packet-dropout process is usually modelled as a Bernoulli process [3], [4], [8] or a Markov chain [4], [9]–[11], and the system is viewed as a special case of jump linear system. In some works [12]–[14], NCSs with arbitrary packet dropouts are modelled as switched systems. The effect of packet delays has also been widely studied. In the works [15]–[17], the NCS is modelled as a time-delay system to tackle the network induced-delay where a state feedback controller is employed. The authors of [18] adopted the common Lyapunov function approach to study the NCS with packet delays and dropouts. In the study [19], the packet delays in the controller to actuator (C/A) channel are treated as the uncertainties of the NCS while the packet dropouts only occur in the sensor to controller (S/C) channel. The authors of [20] employed a fuzzy controller to deal with packet delays in the NCS.

In certain network or system configurations, a multiple-packet transmission policy is required where individual sensor or actuator data are transmitted in separate network packets which may not all arrive at the controller or plant simultaneously due to packet dropouts. By contrast, in a single-packet transmission, all the sensors' or actuators' data are lumped together into one network packet and transmitted at the same time. There are two reasons for adopting multiple-packet transmission. Firstly, a large amount of data must be broken into multiple packets due to the packet size constraint. Secondly and more importantly, sensors and actuators in an NCS may be distributed over a large physical area. There has been some study on the effects of packet dropouts to NCSs under multiple-packet transmission. In [21], the authors give a sufficient condition for stability in scheduling networks where the two packets are alternately sent to the controller, with each of these two packets only carrying partial information of the plant state. In [22], the optimal LQG control problem is considered for two communication channels with packet dropouts. In [9], the authors study stability and controller design of NCSs with packet dropouts driven by a Markov process under the multiple-packet transmission. In [23], the authors analyse the stability of NCSs subject to packet dropouts under the multiple-packet transmission with the packet dropout probability of the communication channel bounded from above.

When the system has parameter uncertainties, the standard $\mathcal{H}_\infty$ control [24] cannot provide guaranteed $\mathcal{H}_\infty$ performance and stability. Robust $\mathcal{H}_\infty$ control has been investigated for both continuous-time and discrete-time systems [10], [25]–[29]. All these references only consider the systems with delays, such as state or network packet delays. To the best of our knowledge, robust $\mathcal{H}_\infty$ control has not been studied for NCSs with packet dropouts under the multiple-packet transmission. The novelty of this contribution is that we study synthesis of robust stochastic stabilisation and design of $\mathcal{H}_\infty$ control for NCSs where the plant has time-varying norm-bounded parameter uncertainties, both the S/C and C/A channels implement multiple-packet transmission policy and experience random packet dropouts. The controller utilises a plant model to estimate the plant state but if any of the multiple packets succeeds in transmission, the controller can replace the corresponding part of the model state with the received partial state information. We formulate this class of NCSs as a stochastic jump linear system. Sufficient conditions are derived for synthesising robust stochastic stabilisation controller and for designing robust $\mathcal{H}_\infty$ controller. These conditions are formulated in the form of linear matrix inequalities (LMIs) that can be solved by the existing numerical techniques [30].

The remainder of this contribution is organised as follows. In Section II, the NCS problem is formulated. Section III addresses the synthesis of robust stochastic stabilisation control and presents an LMI solution, while Section IV considers the robust $\mathcal{H}_\infty$ control design. A numerical example is provided in Section V to illustrate the proposed method, and our conclusions are offered in Section VI. Throughout this contribution we adopt the following notational conventions. $\mathbb{R}$ stands for real numbers and $\mathbb{N}$ for nonnegative integers. $\mathbf{W} > 0$ indicates that $\mathbf{W}$ is a positive-definite matrix. $\mathbf{I}$ and $\mathbf{0}$ represent the identity and zero matrices of appropriate dimensions, respectively. The notation $*$ within a matrix denotes symmetric entries. For a discrete-time signal $\mathbf{w} = \{\mathbf{w}(k)\}_{k\in\mathbb{N}}$ with $\mathbf{w}(k) \in \mathbb{R}^p$, ℓ_2^p denotes the set of $\mathbf{w}$s with $\sum_{k=0}^{\infty} \mathbf{w}^{\mathsf{T}}(k)\mathbf{w}(k) < \infty$.

II. Problem formulation

The NCS $\hat{P}_K$, depicted in Fig. 1, contains a generalised discrete-time plant $\hat{P}$ and a discrete-time controller $\hat{K}$ with the control loop closed via a shared communication network. The plant $\hat{P}$ with parameter uncertainties is described by

$$\begin{cases} \mathbf{x}(k+1) = [\mathbf{A} + \mathbf{\Delta}_A(k)]\mathbf{x}(k) + [\mathbf{B} + \mathbf{\Delta}_B(k)]\mathbf{u}(k) + \mathbf{B}_w\mathbf{w}(k), \\ \mathbf{z}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k), \end{cases} \tag{1}$$

for $\forall k \in \mathbb{N}$, where $\mathbf{x}(k) = [x_1(k) \cdots x_n(k)]^{\mathsf{T}} \in \mathbb{R}^n$, $\mathbf{u}(k) = [u_1(k) \cdots u_m(k)]^{\mathsf{T}} \in \mathbb{R}^m$ and $\mathbf{z}(k) \in \mathbb{R}^q$ are the state, input and controlled output vectors, respectively, $\mathbf{w}(k) \in \mathbb{R}^p$ is the disturbance input vector and $\mathbf{w} \in \ell_2^p$. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{B}_w$, $\mathbf{C}$ and $\mathbf{D}$ are the known constant matrices of appropriate dimensions, while $\mathbf{\Delta}_A(k)$ and $\mathbf{\Delta}_B(k)$ are the unknown matrices representing the time-varying parameter uncertainties which satisfy the following condition

$$[\mathbf{\Delta}_A(k)\ \mathbf{\Delta}_B(k)] = \mathbf{M}\ \mathbf{F}(k)\ [\mathbf{N}_A\ \mathbf{N}_B], \tag{2}$$

where $\mathbf{M}$, $\mathbf{N}_A$ and $\mathbf{N}_B$ are the known constant matrices of appropriate dimensions, while $\mathbf{F}(k)$ is an unknown time-varying matrix with $\mathbf{F}^{\mathsf{T}}(k)\mathbf{F}(k) \leq \mathbf{I}$.

The state and input vectors are transmitted under a multiple-packet transmission policy where at any instant k, the state vector is transmitted by at most n packets and the input vector is transmitted by at most m packets. Network packet dropouts occur in both the S/C and C/A channels. Assume that the n sensors and m actuators are physically distributed. Therefore, n packets are transmitted through the S/C channel at each k, one for each element of $\mathbf{x}(k)$, and similarly m packets are transmitted via the C/A channel at each k, one for each element of $\hat{\mathbf{u}}(k)$. Define $\theta_{s,i}(k) \in \{0, 1\}$ for $i \in \{1, \cdots, n\}$ and $\theta_{a,j}(k) \in \{0, 1\}$ for $j \in \{1, \cdots, m\}$ as the indicators of the single packet dropout in the S/C and C/A channels for $x_i(k)$ and $\hat{u}_j(k)$, respectively, where a value 0 indicates that the packet is dropped while a value 1 indicates that the packet is transmitted successfully. Further define the two matrices of packet dropout indicators as

$$\mathbf{\Theta}_s(k) \triangleq \text{diag}(\theta_{s,1}(k), \theta_{s,2}(k), \ldots, \theta_{s,n}(k)), \tag{3}$$

$$\mathbf{\Theta}_a(k) \triangleq \text{diag}(\theta_{a,1}(k), \theta_{a,2}(k), \ldots, \theta_{a,m}(k)). \tag{4}$$

Remark 1: In our NCS model we mainly consider packet dropouts. This is because most of the present NCSs are configurated over local area networks (LANs), such as wired Ethernet and wireless LAN (WLAN). In such NCSs, packet transmission delay is negligible and the only significant communication delay is due to access delay which is taken into account in our model. Few if any practical NCSs are over wide area networks (WANs). Of course, when considering potential further research of NCSs over WANs, such as control over Internet, packet transmission delay will be significant and cannot be ignored.

The controller $\hat{K}$, similar to the one in [4], consists of the state feedback gain matrix $\mathbf{K} \in \mathbb{R}^{m\times n}$ and the plant model. The controller output is given by

$$\hat{\mathbf{u}}(k) = \mathbf{K}\hat{\mathbf{x}}(k), \tag{5}$$

where $\hat{\mathbf{x}}(k) \in \mathbb{R}^n$ denotes the model state. Referring to Fig. 1, if $\hat{u}_j(k)$ is transmitted successfully through the C/A channel at instant k, $u_j(k) = \hat{u}_j(k)$, otherwise $u_j(k) = 0$. Thus we have

$$\mathbf{u}(k) = \mathbf{\Theta}_a(k)\hat{\mathbf{u}}(k). \tag{6}$$

TCP-like protocol is assumed, in which there is acknowledgement for a received packet. Thus, at each instant k, the network sends an ACK signal to the controller to indicate whether a current control input packet is received or not by the actuator. The plant model is given by

$$\hat{\mathbf{x}}(k+1) = \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{\Theta}_a(k)\hat{\mathbf{u}}(k). \tag{7}$$

If $x_i(k+1)$ is transmitted successfully via the S/C channel at instant $k+1$, the model state variable $\hat{x}_i(k+1)$ is updated by $x_i(k+1)$, otherwise the controller uses the plant model (7) to derive $\hat{x}_i(k+1)$. Thus, we have

$$\begin{aligned} \hat{\mathbf{x}}(k+1) &= \mathbf{\Theta}_s(k+1)\mathbf{x}(k+1) + (\mathbf{I} - \mathbf{\Theta}_s(k+1))(\mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{\Theta}_a(k)\hat{\mathbf{u}}(k)) \\ &= \mathbf{\Theta}_s(k+1)(\mathbf{A} + \mathbf{\Delta}_A(k))\mathbf{x}(k) + \mathbf{\Theta}_s(k+1)\mathbf{B}_w\mathbf{w}(k) \\ &\quad + \Big((\mathbf{I} - \mathbf{\Theta}_s(k+1))\mathbf{A} + (\mathbf{B} + \mathbf{\Theta}_s(k+1)\mathbf{\Delta}_B(k))\mathbf{\Theta}_a(k)\mathbf{K}\Big)\hat{\mathbf{x}}(k). \end{aligned} \tag{8}$$

Define the set

$$\mathcal{S} \triangleq \left\{ (\mathbf{\Omega}_s, \mathbf{\Omega}_a) \,\middle|\, \begin{array}{c} \mathbf{\Omega}_s = \mathrm{diag}\big(\omega_1, \cdots, \omega_n\big),\ \mathbf{\Omega}_a = \mathrm{diag}\big(\omega_{n+1}, \cdots, \omega_{n+m}\big), \\ \omega_i \in \{0,1\}, \forall i \in \{1, \cdots, n+m\}. \end{array} \right\} \tag{9}$$

The number of elements in the set $\mathcal{S}$ is $\bar{r} = 2^{n+m}$. Further define $\mathcal{N} \triangleq \{1, 2, \ldots, \bar{r}\}$ and the mapping f from $\mathcal{S}$ to $\mathcal{N}$:

$$r = f(\mathbf{\Omega}_s, \mathbf{\Omega}_a) = 1 + \sum_{i=1}^{n+m} \omega_i \cdot 2^{i-1}. \tag{10}$$

It is easy to see that f is a one-to-one mapping. In fact, the inverse mapping of f, denoted as

$$(\mathbf{\Omega}_s, \mathbf{\Omega}_a) = (\mathbf{H}_s(r), \mathbf{H}_a(r)), \tag{11}$$

can be implemented by the following iteration algorithm:

- *Step 1*: Set $v = r - 1$, $i = 1$.
- *Step 2*: Find $\tilde{q} \in \mathbb{N}$ and $d \in \{0, 1\}$ to satisfy $v = 2\tilde{q} + d$. Then $\omega_i = d$.
- *Step 3*: If $i < n + m$, then $v = \tilde{q}$, $i = i + 1$, return to *Step 2*.
- *Step 4*: $\mathbf{\Omega}_s = \mathrm{diag}\big(\omega_1, \cdots, \omega_n\big)$, $\mathbf{\Omega}_a = \mathrm{diag}\big(\omega_{n+1}, \cdots, \omega_{n+m}\big)$, *End*.

Thus the sequence $\{(\mathbf{\Theta}_s(k+1), \mathbf{\Theta}_a(k))\}_{k\in\mathbb{N}}$, which specifies the packet dropout process, can be mapped into another sequence $\{r_k\}_{k\in\mathbb{N}}$ with $r_k = f((\mathbf{\Theta}_s(k+1), \mathbf{\Theta}_a(k))$. The inverse mapping of f is simply

$$\left.\begin{aligned} \mathbf{\Theta}_s(k+1) &= \mathbf{H}_s(r_k), \\ \mathbf{\Theta}_a(k) &= \mathbf{H}_a(r_k). \end{aligned}\right\} \tag{12}$$

We now consider the case where $\{r_k\}_{k\in\mathbb{N}}$ is a discrete-time stochastic process.

Assumption 1: r_ks are independently identically distributed (i.i.d.) $\mathcal{N}$-valued random variables. The probability of mass function of r_k is given by $p_i = \text{Prob}(r_k = i)$ with $i \in \mathcal{N}$.

The communication network of Fig. 1 is governed by the (n, m)-packet transmission policy with the associated set $\mathcal{N}$, whose size is $\bar{r} = 2^{n+m}$. This multiple-packet transmission policy is motivated by the fact that in many industrial plants sensors and actuators are distributed over a large physical area and each sensor or actuator has to communicate to the controller individually over the shared network. However, our multiple-packet transmission policy is a generic protocol, as explained in the following remark.

Remark 2: The multiple-packet transmission policy considered in this contribution is a general framework for packet dropouts. At each instant, the number of transmitted packets for the n-dimensional state vector and the m-dimensional input vector are n and m, respectively, for the (n, m)-packet transmission. This multiple-packet transmission policy is actually valid for the case where less than n packets are transmitted in the S/C channel and/or less than m packets are transmitted in the C/A channel, respectively, at each instant. This can simply be achieved by lumping several state or input variables into one packet and by considering the resulting (n', m')-packet transmission scheme, where $n' \le n$ and $m' \le m$. The associated set $\mathcal{N}'$ in this case has a size of $\bar{r}' = 2^{n'+m'}$. For example, assume that $n = 3$ and $m = 1$, and the state variables $x_1(k)$ and $x_2(k)$ are transmitted together in one packet. Then we have the $(2, 1)$-packet transmission policy and the size of $\mathcal{N}'$ is $\bar{r}' = 8$. Let the two packet dropout indicators for the S/C channel be $\theta'_{s,1}(k)$ and $\theta'_{s,2}(k)$. The packet-dropout indicator matrix for the S/C channel takes the form $\mathbf{\Theta}'_s(k) \triangleq \text{diag}(\theta'_{s,1}(k), \theta'_{s,1}(k), \theta'_{s,2}(k))$.

Define the state of the NCS $\hat{P}_K$ as

$$\overline{\mathbf{x}}(k) \triangleq [\mathbf{x}^{\mathsf{T}}(k)\ \mathbf{e}^{\mathsf{T}}(k)]^{\mathsf{T}}, \tag{13}$$

where $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$. From (1) and (8), the NCS $\hat{P}_K$ can be described by

$$\begin{bmatrix} \overline{\mathbf{x}}(k+1) \\ \mathbf{z}(k) \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{A}}_{r_k} & \overline{\mathbf{B}}_{r_k} \\ \overline{\mathbf{C}}_{r_k} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \overline{\mathbf{x}}(k) \\ \mathbf{w}(k) \end{bmatrix}, \ r_k \in \mathcal{N}, \tag{14}$$

where

$$\overline{\mathbf{A}}_{r_k} = \begin{bmatrix} \mathbf{A} + \mathbf{\Delta}_A(k) + (\mathbf{B} + \mathbf{\Delta}_B(k))\mathbf{\Theta}_a(k)\mathbf{K} & -(\mathbf{B} + \mathbf{\Delta}_B(k))\mathbf{\Theta}_a(k)\mathbf{K} \\ (\mathbf{I} - \mathbf{\Theta}_s(k+1))(\mathbf{\Delta}_A(k) + \mathbf{\Delta}_B(k)\mathbf{\Theta}_a(k)\mathbf{K}) & (\mathbf{I} - \mathbf{\Theta}_s(k+1))(\mathbf{A} - \mathbf{\Delta}_B(k)\mathbf{\Theta}_a(k)\mathbf{K}) \end{bmatrix}, \tag{15}$$

$$\overline{\mathbf{B}}_{r_k} = \begin{bmatrix} \mathbf{B}_w \\ (\mathbf{I} - \mathbf{\Theta}_s(k+1))\mathbf{B}_w \end{bmatrix}, \tag{16}$$

$$\overline{\mathbf{C}}_{r_k} = \begin{bmatrix} \mathbf{C} + \mathbf{D}\mathbf{\Theta}_a(k)\mathbf{K} & -\mathbf{D}\mathbf{\Theta}_a(k)\mathbf{K} \end{bmatrix}, \tag{17}$$

while $\mathbf{\Theta}_s(k+1)$ and $\mathbf{\Theta}_a(k)$ are given in (3) and (4). From (2) and (10), $\overline{\mathbf{A}}_{r_k}$ can be written as $\overline{\mathbf{A}}_i = \mathbf{\Phi}_i + \overline{\mathbf{M}}_i \mathbf{F}(k)\mathbf{\Gamma}_i$ for $i \in \mathcal{N}$, where

$$\mathbf{\Phi}_i = \begin{bmatrix} \mathbf{A} + \mathbf{B}\mathbf{H}_a(i)\mathbf{K} & -\mathbf{B}\mathbf{H}_a(i)\mathbf{K} \\ \mathbf{0} & (\mathbf{I} - \mathbf{H}_s(i))\mathbf{A} \end{bmatrix}, \tag{18}$$

$$\mathbf{\Gamma}_i = \begin{bmatrix} \mathbf{N}_A + \mathbf{N}_B\mathbf{H}_a(i)\mathbf{K} & -\mathbf{N}_B\mathbf{H}_a(i)\mathbf{K} \end{bmatrix}, \tag{19}$$

$$\overline{\mathbf{M}}_i = \begin{bmatrix} \mathbf{M} \\ (\mathbf{I} - \mathbf{H}_s(i))\mathbf{M} \end{bmatrix}, \tag{20}$$

with $\mathbf{H}_a(i)$ and $\mathbf{H}_s(i)$ given in (12). We introduce the following concepts of robust stochastic stability and robust $\mathcal{H}_\infty$ performance for the NCS $\hat{P}_K$.

Definition 1: (See [10], [27]) The NCS $\hat{P}_K$ with $\mathbf{w}(k) \equiv \mathbf{0}$ is said to be *robustly stochastically stable* if for any initial condition $\overline{\mathbf{x}}(0) \in \mathbb{R}^{2n}$,

$$\sum_{k=0}^{\infty} \mathsf{E}\left[\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)\right] < \infty \tag{21}$$

holds for all the admissible uncertainties $\mathbf{\Delta}_A(k)$ and $\mathbf{\Delta}_B(k)$, where $\mathsf{E}[\cdot]$ denotes the expectation.

Definition 2: (See [10], [27]) The NCS $\hat{P}_K$ is said to be *robustly stochastically stable with disturbance attenuation level* $\gamma > 0$ if $\hat{P}_K$ with $\mathbf{w}(k) \equiv \mathbf{0}$ is robustly stochastically stable and, for any nonzero $\mathbf{w} \in \ell_2^p$, the response $\{\mathbf{z}(k)\}_{k\in\mathbb{N}}$ under the zero initial condition $\overline{\mathbf{x}}(0) = \mathbf{0}$ satisfies

$$\sum_{k=0}^{\infty} \mathsf{E}\Big[\mathbf{z}^{\mathsf{T}}(k)\mathbf{z}(k)\Big] < \gamma^2 \Big[\sum_{k=0}^{\infty} \mathbf{w}^{\mathsf{T}}(k)\mathbf{w}(k)\Big]. \tag{22}$$

III. ROBUST STABILISATION

The task of stabilisation control is as follows. Given $\mathbf{A}$, $\mathbf{B}$, $\mathbf{N}_A$, $\mathbf{N}_B$ and $\mathbf{M}$ as well as the chosen multiple-packet transmission policy with *Assumption* 1, determine the controller $\mathbf{K}$ such that the NCS $\hat{P}_K$ is robustly stochastically stable.

The following lemma from [31] is useful for the proofs of our main results.

Lemma 1: Let $\mathbf{Z}$, $\mathbf{U}$, $\mathbf{H}$, $\mathbf{G}$ and $\tilde{\mathbf{F}}$ be the real matrices of appropriate dimensions such that $\mathbf{G} > 0$ and $\tilde{\mathbf{F}}^{\mathsf{T}}\tilde{\mathbf{F}} \leq \mathbf{I}$. Then, for any scalar $\epsilon > 0$ such that $\mathbf{G} - \epsilon\mathbf{U}\mathbf{U}^{\mathsf{T}} > 0$, we have

$$(\mathbf{Z} + \mathbf{U}\tilde{\mathbf{F}}\mathbf{H})^{\mathsf{T}}\mathbf{G}^{-1}(\mathbf{Z} + \mathbf{U}\tilde{\mathbf{F}}\mathbf{H}) \leq \mathbf{Z}^{\mathsf{T}}(\mathbf{G} - \epsilon\mathbf{U}\mathbf{U}^{\mathsf{T}})^{-1}\mathbf{Z} + \epsilon^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{H}. \tag{23}$$

Theorem 1: For the NCS $\hat{P}_K$ with $\mathbf{w}(k) \equiv \mathbf{0}$ under *Assumption* 1, suppose that there exist scalars $\epsilon_i > 0$ with $i \in \mathcal{N}$, matrices $\mathbf{Q} > 0$ and $\mathbf{Y}$ such that the following LMI is satisfied:

$$\begin{bmatrix} -\tilde{\mathbf{Q}} & * & * & \cdots & * \\ \tilde{\mathbf{\Pi}}_1 & \mathbf{\Upsilon}_1 & * & \cdots & * \\ \tilde{\mathbf{\Pi}}_2 & \mathbf{0} & \mathbf{\Upsilon}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ \tilde{\mathbf{\Pi}}_{\bar{r}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{\Upsilon}_{\bar{r}} \end{bmatrix} \triangleq \tilde{\mathbf{\Lambda}} < 0, \tag{24}$$

where for $i \in \mathcal{N}$

$$\tilde{\mathbf{Q}} = \mathrm{diag}(\mathbf{Q}, \mathbf{Q}), \tag{25}$$

$$\tilde{\mathbf{\Pi}}_i = \sqrt{p_i}[\tilde{\mathbf{\Phi}}_i^{\mathsf{T}} \ \ \tilde{\mathbf{\Gamma}}_i^{\mathsf{T}}]^{\mathsf{T}}, \tag{26}$$

$$\mathbf{\Upsilon}_i = \mathrm{diag}\left(\epsilon_i\overline{\mathbf{M}}_i\overline{\mathbf{M}}_i^{\mathsf{T}} - \tilde{\mathbf{Q}}, \ -\epsilon_i\mathbf{I}\right), \tag{27}$$

$$\tilde{\mathbf{\Phi}}_i = \begin{bmatrix} \mathbf{A}\mathbf{Q} + \mathbf{B}\mathbf{H}_a(i)\mathbf{Y} & -\mathbf{B}\mathbf{H}_a(i)\mathbf{Y} \\ \mathbf{0} & (\mathbf{I} - \mathbf{H}_s(i))\mathbf{A}\mathbf{Q} \end{bmatrix}, \tag{28}$$

$$\tilde{\mathbf{\Gamma}}_i = \begin{bmatrix} \mathbf{N}_A\mathbf{Q} + \mathbf{N}_B\mathbf{H}_a(i)\mathbf{Y} & -\mathbf{N}_B\mathbf{H}_a(i)\mathbf{Y} \end{bmatrix}, \tag{29}$$

while $\overline{\mathbf{M}}_i$ is defined in (20). Then $\mathbf{K} = \mathbf{Y}\mathbf{Q}^{-1}$ makes $\hat{P}_K$ robustly stochastically stable.

Proof Let $\mathbf{P} = \mathbf{Q}^{-1}$, then $\tilde{\mathbf{P}} = \tilde{\mathbf{Q}}^{-1}$. From (24), it is easy to show that

$$\mathbf{\Psi}_i \triangleq \tilde{\mathbf{P}}^{-1} - \epsilon_i\overline{\mathbf{M}}_i\overline{\mathbf{M}}_i^{\mathsf{T}} > 0, \ \forall i \in \mathcal{N}. \tag{30}$$

Now for the NCS $\hat{P}_K$, construct the Lyapunov function

$$V(k) \triangleq \overline{\mathbf{x}}^{\mathsf{T}}(k)\tilde{\mathbf{P}}\overline{\mathbf{x}}(k), \ \forall k \in \mathbb{N}. \tag{31}$$

Noticing $\epsilon_i > 0$, (20) and (30) as well as using *Lemma* 1, we have

$$\begin{aligned} \mathsf{E}[V(k+1)] - V(k) &= \overline{\mathbf{x}}^{\mathsf{T}}(k)\Big[\sum_{i\in\mathcal{N}} p_i \overline{\mathbf{A}}_i^{\mathsf{T}} \tilde{\mathbf{P}} \overline{\mathbf{A}}_i - \tilde{\mathbf{P}}\Big]\overline{\mathbf{x}}(k) \\ &= \overline{\mathbf{x}}^{\mathsf{T}}(k)\Big[\sum_{i\in\mathcal{N}} p_i(\mathbf{\Phi}_i + \overline{\mathbf{M}}_i\mathbf{F}(k)\mathbf{\Gamma}_i)^{\mathsf{T}}\tilde{\mathbf{P}}(\mathbf{\Phi}_i + \overline{\mathbf{M}}_i\mathbf{F}(k)\mathbf{\Gamma}_i) - \tilde{\mathbf{P}}\Big]\overline{\mathbf{x}}(k) \\ &\leq \overline{\mathbf{x}}^{\mathsf{T}}(k)\mathbf{\Lambda}\overline{\mathbf{x}}(k), \end{aligned} \tag{32}$$

where

$$\mathbf{\Lambda} = \sum_{i\in\mathcal{N}} p_i\big(\mathbf{\Phi}_i^{\mathsf{T}}\mathbf{\Psi}_i^{-1}\mathbf{\Phi}_i + \epsilon_i^{-1}\mathbf{\Gamma}_i^{\mathsf{T}}\mathbf{\Gamma}_i\big) - \tilde{\mathbf{P}}. \tag{33}$$

On the other hand, pre- and post-multiplying (24) by diag$(\tilde{\mathbf{P}}, \mathbf{I})$ yields

$$\begin{bmatrix} -\tilde{\mathbf{P}} & * & * & \cdots & * \\ \mathbf{\Pi}_1 & \mathbf{\Upsilon}_1 & * & \cdots & * \\ \mathbf{\Pi}_2 & \mathbf{0} & \mathbf{\Upsilon}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ \mathbf{\Pi}_{\bar{r}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{\Upsilon}_{\bar{r}} \end{bmatrix} < 0, \tag{34}$$

where for $i \in \mathcal{N}$

$$\mathbf{\Pi}_i = \sqrt{p_i}[\mathbf{\Phi}_i^{\mathsf{T}} \ \mathbf{\Gamma}_i^{\mathsf{T}}]^{\mathsf{T}}, \tag{35}$$

while $\mathbf{\Phi}_i$ and $\mathbf{\Gamma}_i$ are given in (18) and (19), respectively. By Schur complement, (34) implies that $\mathbf{\Lambda} < 0$. This together with (32) leads to

$$\mathsf{E}[V(k+1)] - V(k) \leq -\tau\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k), \tag{36}$$

where $\tau = \lambda_{\min}(-\mathbf{\Lambda})$ denotes the minimal eigenvalue of $-\mathbf{\Lambda}$. From (36), we obtain

$$\mathsf{E}[V(T+1) - V(0)] = \sum_{k=0}^{T} \mathsf{E}\big[\mathsf{E}[V(k+1)] - V(k)\big] \leq -\tau \sum_{k=0}^{T} \mathsf{E}[\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)] \tag{37}$$

for any $T \geq 1$, which implies

$$\sum_{k=0}^{T} \mathsf{E}[\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)] \leq \frac{1}{\tau}\big(\mathsf{E}[V(0)] - \mathsf{E}[V(T+1)]\big) \leq \frac{1}{\tau}V(0) < \infty. \tag{38}$$

According to *Definition* 1, the NCS $\hat{P}_K$ with $\mathbf{w}(k) \equiv \mathbf{0}$ is robustly stochastically stable. ∎

IV. Robust $\mathcal{H}_\infty$ Control

The task of designing robust $\mathcal{H}_\infty$ controller is as follows. Given $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{B}_w$, $\mathbf{N}_A$, $\mathbf{N}_B$, $\mathbf{M}$ and $\gamma > 0$ as well as the specified multiple-packet transmission policy with *Assumption* 1, determine the controller $\mathbf{K}$ such that the NCS $\hat{P}_K$ is robustly stochastically stable with the specified disturbance attenuation level γ.

A sufficient condition is proposed for designing robust $\mathcal{H}_\infty$ controller, and our main result is given in the following theorem.

Theorem 2: Given a scalar $\gamma > 0$, the NCS $\hat{P}_K$ under *Assumption* 1 is robustly stochastically stable with disturbance attenuation level γ, if there exist scalars $\epsilon_i > 0$ for $i \in \mathcal{N}$, matrices $\mathbf{Q} > 0$ and $\mathbf{Y}$ such that the following LMI is satisfied:

$$\begin{bmatrix} \begin{bmatrix} -\tilde{\mathbf{Q}} & \mathbf{0} \\ \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix} & * & * & \cdots & * \\ \mathbf{\Omega}_1 & \mathbf{\Xi}_1 & * & \cdots & * \\ \mathbf{\Omega}_2 & \mathbf{0} & \mathbf{\Xi}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ \mathbf{\Omega}_{\bar{r}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{\Xi}_{\bar{r}} \end{bmatrix} < 0, \tag{39}$$

where for $i \in \mathcal{N}$,

$$\mathbf{\Omega}_i = \sqrt{p_i} \begin{bmatrix} \tilde{\mathbf{\Phi}}_i & \overline{\mathbf{B}}_i \\ \tilde{\mathbf{\Gamma}}_i & \mathbf{0} \\ \tilde{\mathbf{C}}_i & \mathbf{0} \end{bmatrix}, \tag{40}$$

$$\mathbf{\Xi}_i = \mathrm{diag}\left(\epsilon_i \overline{\mathbf{M}}_i \overline{\mathbf{M}}_i^{\mathsf{T}} - \tilde{\mathbf{Q}},\ -\epsilon_i\mathbf{I},\ -\mathbf{I}\right), \tag{41}$$

$$\tilde{\mathbf{C}}_i = \begin{bmatrix} \mathbf{C}\mathbf{Q} + \mathbf{D}\mathbf{H}_a(i)\mathbf{Y} & -\mathbf{D}\mathbf{H}_a(i)\mathbf{Y} \end{bmatrix}, \tag{42}$$

while $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{\Phi}}_i$, $\tilde{\mathbf{\Gamma}}_i$, $\overline{\mathbf{B}}_i$ and $\overline{\mathbf{M}}_i$ are given in (25), (28), (29), (16) and (20), respectively. In this case, the state feedback gain matrix is given by $\mathbf{K} = \mathbf{Y}\mathbf{Q}^{-1}$.

Proof From (39), we can directly obtain

$$\tilde{\mathbf{\Lambda}} \le \tilde{\mathbf{\Lambda}} + \sum_{i\in\mathcal{N}} \begin{bmatrix} \tilde{\mathbf{C}}_i^{\mathsf{T}} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{C}}_i & \mathbf{0} \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{B}}^{\mathsf{T}} \end{bmatrix} < 0. \tag{43}$$

where

$$\tilde{\mathbf{B}}^{\mathsf{T}} = \begin{bmatrix} \overline{\mathbf{B}}_1^{\mathsf{T}} & \mathbf{0} & \overline{\mathbf{B}}_2^{\mathsf{T}} & \mathbf{0} & \cdots & \overline{\mathbf{B}}_{\bar{r}}^{\mathsf{T}} & \mathbf{0} \end{bmatrix}, \tag{44}$$

while $\tilde{\mathbf{\Lambda}}$ is defined in (24). Therefore, it follows from *Theorem* 1 that the NCS $\hat{P}_K$ with $\mathbf{w}(k) \equiv \mathbf{0}$ is robustly stochastically stable.

Next, we prove that the NCS $\hat{P}_K$ has the required noise attenuation level γ for any nonzero $\mathbf{w} \in \ell_2^p$. Let $\mathbf{P} = \mathbf{Q}^{-1}$, then $\tilde{\mathbf{P}} = \tilde{\mathbf{Q}}^{-1}$. Consider the Lyapunov function $V(k)$ defined in (31) with the zero initial condition $\overline{\mathbf{x}}(0) = \mathbf{0}$ and $V(0) = 0$. It follows from (37) that for any $T \geq 1$

$$\sum_{k=0}^{T} \mathsf{E}\big[\mathsf{E}[V(k+1)] - V(k)\big] = \mathsf{E}[V(T+1)] \geq 0. \tag{45}$$

Since $\epsilon_i > 0$ for $i \in \mathcal{N}$ and (30) is satisfied due to (39), according to *Lemma* 1 we have

$$\mathsf{E}[V(k+1)] = \big[\overline{\mathbf{x}}^\mathsf{T}(k)\ \mathbf{w}^\mathsf{T}(k)\big]\mathbf{S}\big[\overline{\mathbf{x}}^\mathsf{T}(k)\ \mathbf{w}^\mathsf{T}(k)\big]^\mathsf{T} \leq \big[\overline{\mathbf{x}}^\mathsf{T}(k)\ \mathbf{w}^\mathsf{T}(k)\big]\hat{\mathbf{S}}\big[\overline{\mathbf{x}}^\mathsf{T}(k)\ \mathbf{w}^\mathsf{T}(k)\big]^\mathsf{T}, \tag{46}$$

where

$$\begin{aligned} \mathbf{S} &= \sum_{i\in\mathcal{N}} p_i \big[\overline{\mathbf{A}}_i\ \overline{\mathbf{B}}_i\big]^\mathsf{T} \tilde{\mathbf{P}} \big[\overline{\mathbf{A}}_i\ \overline{\mathbf{B}}_i\big] \\ &= \sum_{i\in\mathcal{N}} p_i \left(\big[\mathbf{\Phi}_i\ \overline{\mathbf{B}}_i\big] + \overline{\mathbf{M}}_i \mathbf{F}(k)\big[\mathbf{\Gamma}_i\ \mathbf{0}\big]\right)^\mathsf{T} \tilde{\mathbf{P}} \left(\big[\mathbf{\Phi}_i\ \overline{\mathbf{B}}_i\big] + \overline{\mathbf{M}}_i \mathbf{F}(k)\big[\mathbf{\Gamma}_i\ \mathbf{0}\big]\right), \end{aligned} \tag{47}$$

$$\hat{\mathbf{S}} = \sum_{i\in\mathcal{N}} p_i \Big(\begin{bmatrix} \mathbf{\Phi}_i^\mathsf{T} \\ \overline{\mathbf{B}}_i^\mathsf{T} \end{bmatrix} \mathbf{\Psi}_i^{-1} \begin{bmatrix} \mathbf{\Phi}_i & \overline{\mathbf{B}}_i \end{bmatrix} + \epsilon_i^{-1} \begin{bmatrix} \mathbf{\Gamma}_i^\mathsf{T} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma}_i & \mathbf{0} \end{bmatrix} \Big). \tag{48}$$

Combining (14) and (46) yields

$$\mathsf{E}\left[V(k+1)\right] - V(k) + \mathbf{z}^\mathsf{T}(k)\mathbf{z}(k) - \gamma^2 \mathbf{w}^\mathsf{T}(k)\mathbf{w}(k) \leq \big[\overline{\mathbf{x}}^\mathsf{T}(k)\ \mathbf{w}^\mathsf{T}(k)\big]\hat{\mathbf{\Lambda}}\big[\overline{\mathbf{x}}^\mathsf{T}(k)\ \mathbf{w}^\mathsf{T}(k)\big]^\mathsf{T}, \tag{49}$$

where

$$\begin{aligned} \hat{\mathbf{\Lambda}} &= \sum_{i\in\mathcal{N}} p_i \left(\hat{\mathbf{S}} + \begin{bmatrix} \overline{\mathbf{C}}_i^\mathsf{T} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \overline{\mathbf{C}}_i & \mathbf{0} \end{bmatrix} \right) - \begin{bmatrix} \tilde{\mathbf{P}} & \mathbf{0} \\ \mathbf{0} & \gamma^2 \mathbf{I} \end{bmatrix} \\ &= \sum_{i\in\mathcal{N}} p_i \begin{bmatrix} \mathbf{\Phi}_i & \overline{\mathbf{B}}_i \\ \mathbf{\Gamma}_i & \mathbf{0} \\ \overline{\mathbf{C}}_i & \mathbf{0} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbf{\Psi}_i^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \epsilon_i^{-1}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_i & \overline{\mathbf{B}}_i \\ \mathbf{\Gamma}_i & \mathbf{0} \\ \overline{\mathbf{C}}_i & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{P}} & \mathbf{0} \\ \mathbf{0} & \gamma^2 \mathbf{I} \end{bmatrix}, \end{aligned} \tag{50}$$

and $\mathbf{\Psi}_i$ is defined in (30). On the other hand, pre- and post-multiplying (39) by diag$(\tilde{\mathbf{P}}, \mathbf{I})$ as well as applying Schur complement yields

$$\hat{\mathbf{\Lambda}} < 0. \tag{51}$$

Let us define the performance function

$$J(T) = \sum_{k=0}^{T} \mathsf{E}\Big[\mathbf{z}^{\mathsf{T}}(k)\mathbf{z}(k) - \gamma^2 \mathbf{w}^{\mathsf{T}}(k)\mathbf{w}(k)\Big]. \tag{52}$$

Then from (45), (49) and (52), we derive

$$\begin{aligned} J(T) &= \sum_{k=0}^{T} \mathsf{E}\Big[\big(\mathbf{z}^{\mathsf{T}}(k)\mathbf{z}(k) - \gamma^2 \mathbf{w}^{\mathsf{T}}(k)\mathbf{w}(k) + V(k+1) - V(k)\big) - \big(V(k+1) - V(k)\big)\Big] \\ &\leq \sum_{k=0}^{T} \mathsf{E}\Big[\big[\overline{\mathbf{x}}^{\mathsf{T}}(k)\ \mathbf{w}^{\mathsf{T}}(k)\big]\hat{\mathbf{\Lambda}}\big[\overline{\mathbf{x}}^{\mathsf{T}}(k)\ \mathbf{w}^{\mathsf{T}}(k)\big]^{\mathsf{T}}\Big] - \mathsf{E}[V(T+1)]. \end{aligned} \tag{53}$$

$\forall \mathbf{w}(k) \neq \mathbf{0}$, (51) and (53) lead to $J(\infty) < 0$. This completes the proof of *Theorem* 2. ■

Remark 3: Theorems 1 and 2 provide sufficient conditions for robust stabilisation and robust $\mathcal{H}_\infty$ control, respectively, for NCSs with uncertainties and multiple-packet transmission. These results were not seen previously in the existing literature. The main advantage of our approach is that our results are valid for the generic case where random packet dropouts occur in the multiple (S/C and C/A) channels independently and the plant has uncertainties. This should be contrasted with the existing works [10], [15]–[19], [25]–[29], which only consider the NCSs with delays and single-packet transmission. In particular, although the work of [19] also employs a plant model, it assumes that packet dropouts never occur in the C/A channel and the uncertainties are not about the plant but are associated with the packet delays in the C/A channel. In reality, packet dropouts can occur independently in the C/A channel, which must be taken into account as our approach does.

V. A Numerical Example

To illustrate the effectiveness of the proposed approach, we considered the uncertain NCS $\hat{P}_K$ of $\mathbf{x}(k) \in \mathbb{R}^3$, $\mathbf{u}(k) \in \mathbb{R}^2$, $\mathbf{z}(k) \in \mathbb{R}$ and $\mathbf{w}(k) \in \mathbb{R}$, with the following plant parameters

$$\mathbf{A} = \begin{bmatrix} -0.2 & 0 & 0.9 \\ 0.6 & -0.9 & 0.5 \\ 0.2 & -1 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0.2 & 0.4 \\ 0.9 & 0.8 \\ 0.3 & 0.7 \end{bmatrix}, \tag{54}$$

$$\mathbf{N}_A = \begin{bmatrix} 0.6 & 0.2 & 0.7 \end{bmatrix}, \ \mathbf{N}_B = \begin{bmatrix} 0.5 & 0.8 \end{bmatrix}, \ \mathbf{M} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.2 \end{bmatrix}, \tag{55}$$

$$\mathbf{B}_w = \begin{bmatrix} 0.1 \\ 0.1 \\ -0.2 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 0.2 & 0.3 & 0.3 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 0.7 & 0.9 \end{bmatrix}. \tag{56}$$

The eigenvalues of the plant were -1.0554 and $-0.0233 \pm 0.6726\ i$.

Synthesising robust stabilisation control

We considered the uncertain NCS $\hat{P}_K$ with the plant parameters $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{N}_A$ and $\mathbf{N}_B$ given in (54) and (55). At each instant k, the state vector $\mathbf{x}(k)$ was transmitted by two packets through the S/C channel, with $x_1(k)$ and $x_2(k)$ lumped in one packet while $x_3(k)$ in the other packet, and the input vector $\hat{\mathbf{u}}(k)$ was transmitted by two packets via the C/A channel. Thus the $(2,2)$-packet transmission was implemented for the NCS $\hat{P}_K$ (see *Remark* 2), and the number of elements in the set $\mathcal{N}'$ was $\bar{r}' = 2^{2+2} = 16$. Without the loss of generality, $p_i = 1/16$ was assumed for $i \in \mathcal{N}'$. This implied that the packet dropout rate was 50% for any packet. Our objective was to design the state feedback gain matrix $\mathbf{K}$ such that, for all the admissible uncertainties, the NCS $\hat{P}_K$ with $\mathbf{w}(k) \equiv \mathbf{0}$ was robustly stochastically stable.

By applying the Matlab LMI Control Toolbox to solve the LMI (24) we obtained the following solution

$$\mathbf{Q} = \begin{bmatrix} 25.1254 & 7.8509 & -5.9615 \\ 7.8509 & 11.5817 & 1.6578 \\ -5.9615 & 1.6578 & 18.4273 \end{bmatrix}, \ \mathbf{Y} = \begin{bmatrix} -4.5417 & 1.2570 & 2.8069 \\ 0.6289 & 1.4668 & -2.7677 \end{bmatrix},$$

$$\epsilon_1 = 22.9428,\ \epsilon_2 = 23.3998,\ \epsilon_3 = 25.2530,\ \epsilon_4 = 27.1442,\ \epsilon_5 = 24.8060,\ \epsilon_6 = 25.2977,$$

$$\epsilon_7 = 27.3540,\ \epsilon_8 = 29.4228,\ \epsilon_9 = 22.2171,\ \epsilon_{10} = 22.6745,\ \epsilon_{11} = 24.4595,\ \epsilon_{12} = 26.3249,$$

$$\epsilon_{13} = 24.0862,\ \epsilon_{14} = 24.5757,\ \epsilon_{15} = 26.5540,\ \epsilon_{16} = 28.5913.$$

It followed from *Theorem* 1 that the robust stochastic stabilisation control problem was solvable with the state feedback gain matrix given by

$$\mathbf{K} = \mathbf{Y}\mathbf{Q}^{-1} = \begin{bmatrix} -0.2564 & 0.2760 & 0.0445 \\ -0.0895 & 0.2157 & -0.1986 \end{bmatrix}.$$

This designed controller $\mathbf{K}$ was then used in the following simulation, where the initial state was chosen to be

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{e}(0) = \begin{bmatrix} e_1(0) \\ e_2(0) \\ e_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}. \tag{57}$$

The disturbance $\mathbf{w}(k)$ was assumed to be uniformly distributed within $[-0.1,\ 0.1]$ for the interval $k \in [0,\ 100]$ and zero elsewhere. Fig. 2 (a) and (b) depict typical response of the state trajectories for $\mathbf{x}(k)$ and the error trajectories for $\mathbf{e}(k)$, respectively. The NCS was simulated 500 times with the same initial condition (57) and the same packet dropout rates. For any $k \in \mathbb{N}$, we obtained 500 observations of the random variable $\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)$. The first sample moment of the observations, denoted by $\mathsf{E}_e\left[\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)\right]$, was computed. According to the standard statistics theory, $\mathsf{E}_e\left[\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)\right]$ is a confident estimation of $\mathsf{E}\left[\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)\right]$ when the observation number is large. Fig. 3 depicts the trajectory of $\mathsf{E}_e\left[\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)\right]$ where as expected it can be seen that $\mathsf{E}_e\left[\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)\right]$ converged to zero.

Designing robust $\mathcal{H}_\infty$ control

We considered the uncertain NCS $\hat{P}_K$ with the plant parameters $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{N}_A$, $\mathbf{N}_B$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{B}_w$ given in (54) to (56). At each instant k, the state vector $\mathbf{x}(k)$ was transmitted by three packets through the S/C channel and the input vector $\hat{\mathbf{u}}(k)$ was transmitted by two packets via the C/A channel. Thus the $(n = 3, m = 2)$-packet transmission was implemented, and the number of elements in the set $\mathcal{N}$ was $\bar{r} = 2^{3+2} = 32$. We assumed $p_i = 1/32$ for $i \in \mathcal{N}$, which implied that the packet dropout rate was 50% for any state or input variable. Our objective was to design the state feedback gain matrix $\mathbf{K}$ such that, for all the admissible uncertainties, the NCS $\hat{P}_K$ was robustly stochastically stable with the specified disturbance attenuation level $\gamma > 0$.

Assuming $\gamma = 0.5$, we applied the Matlab LMI Control Toolbox to solve the LMI (39) and obtained the following solution

$$\mathbf{Q} = \begin{bmatrix} 1.5324 & 0.5912 & -0.3228 \\ 0.5912 & 0.6050 & 0.0328 \\ -0.3228 & 0.0328 & 1.0706 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -0.4570 & -0.0826 & 0.1417 \\ 0.1029 & 0.0776 & -0.0903 \end{bmatrix},$$

$$\epsilon_1 = 1.9038,\ \epsilon_2 = 2.5198,\ \epsilon_3 = 1.8389,\ \epsilon_4 = 3.4124,\ \epsilon_5 = 1.7918,\ \epsilon_6 = 1.9885,$$

$$\epsilon_7 = 2.2131,\ \epsilon_8 = 4.3420,\ \epsilon_9 = 1.9587,\ \epsilon_{10} = 2.5863,\ \epsilon_{11} = 1.8941,\ \epsilon_{12} = 3.5462,$$

$$\epsilon_{13} = 1.8370,\ \epsilon_{14} = 2.0316,\ \epsilon_{15} = 2.2882,\ \epsilon_{16} = 4.5561,\ \epsilon_{17} = 1.9077,\ \epsilon_{18} = 2.5370,$$

$$\epsilon_{19} = 1.8372,\ \epsilon_{20} = 3.3446,\ \epsilon_{21} = 1.8092,\ \epsilon_{22} = 2.0223,\ \epsilon_{23} = 2.1991,\ \epsilon_{24} = 4.1983,$$

$$\epsilon_{25} = 1.9575,\ \epsilon_{26} = 2.6013,\ \epsilon_{27} = 1.8873,\ \epsilon_{28} = 3.4879,$$

$$\epsilon_{29} = 1.8487,\ \epsilon_{30} = 2.0596,\ \epsilon_{31} = 2.2694,\ \epsilon_{32} = 4.4325.$$

It followed from *Theorem* 2 that the robust $\mathcal{H}_\infty$ control problem was solvable with the state feedback gain matrix given by

$$\mathbf{K} = \mathbf{Y}\mathbf{Q}^{-1} = \begin{bmatrix} -0.3916 & 0.2458 & 0.0067 \\ -0.0053 & 0.1384 & -0.0902 \end{bmatrix}.$$

The above NCS example clearly demonstrates that our approach can effectively design the controller to satisfy the stochastic stability and the required $\mathcal{H}_\infty$ performance criterion.

VI. Conclusions

In this contribution we have investigated a class of NCSs where the plant has time-varying norm-bounded parameter uncertainties, both the S/C and C/A channels implement multiple-packet transmission scheme and impose random packet dropouts. Firstly we have established sufficient conditions in the form of LMI for synthesising robust stochastic stabilisation controller. Secondly we have considered the robust $\mathcal{H}_\infty$ controller design and have presented the LMI solution for robust $\mathcal{H}_\infty$ control law that stabilises this class of NCSs with a prescribed disturbance attenuation level. A numerical example has been included to illustrate our proposed design approach.

References

[1] Moyne, J.R., and Tilbury, D.M.: 'The emergence of industrial control networks for manufacturing control, diagnostics, and safety data,' *Proc. IEEE*, 2007, 95, (1), pp.29–47

[2] Hespanha, J.P., Naghshtabrizi, P., and Xu, Y.-G.: 'A survey of recent results in networked control systems,' *Proc. IEEE*, 2007, 95, (1), pp.138–162

[3] Schenato, L., Sinopoli, B., Franceschetti, M., Poolla, K., and Sastry, S.S.: 'Foundations of control and estimation over lossy networks,' *Proc. IEEE*, 2007, 95, (1), pp.163–187

[4] Montestruque, L.A., and Antsaklis, P.J.: 'Stability of model-based networked control systems with time-varying transmission times,' *IEEE Trans. Automatic Control*, 2004, 49, (9), pp.1562–1572

[5] Mao, Z., and Jiang, B.: ‘Fault identification and fault-tolerant control for a class of networked control systems,’ *Int. J. Innovative Computing, Information and Control*, 2007, 3, (5), pp.1121–1130

[6] Ji, Y., Chizeck, H.J., Feng, X., and Loparo, K.A.: ‘Stability and control of discrete-time jump linear systems,’ *Control Theory and Advanced Technology*, 1991, 7, (2), pp.247–270

[7] Costa, O.L.V., and Fragoso, M.D.: ‘Stability results for discrete-time linear systems with Markovian jumping parameters,’ *J. Mathematical Analysis and Applications*, 1993, 179, (1), pp.154–178

[8] Imer, O.C., Yüksel, S., and Başar, T.: ‘Optimal control of LTI systems over unreliable communication links,’ *Automatica*, 2006, 42, (9), pp.1429–1439

[9] Wu, J., and Chen, T.: ‘Design of networked control systems with packet dropouts,’ *IEEE Trans. Automatic Control*, 2007, 52, (7), pp.1314–1319

[10] Boukas, E.K., and Liu, Z.K.: ‘Robust H_∞ control of discrete-time Markovian jump linear systems with mode-dependent time-delays,’ *IEEE Trans. Automatic Control*, 2001, 46, (12), pp.1918–1924

[11] Seiler, P., and Sengupta, R.: ‘An H_∞ approach to networked control,’ *IEEE Trans. Automatic Control*, 2005, 50, (3), pp.356–364

[12] Yu, M., Wang, L., Chu, T., and Xie, G.: ‘Stabilization of networked control systems with data packet dropout and network delays via switching system approach,’ *Proc. 43rd IEEE Conf. Decision and Control*, 2004, pp.3539–3544

[13] Zhang, W.-A., and Yu, L.: ‘Output feedback stabilization of networked control systems with packet dropouts,’ *IEEE Trans. Automatic Control*, 2007, 52, (9), pp.1705–1710

[14] Ishii, H.: ‘$\mathcal{H}_\infty$ control with limited communication and message losses,’ *System and Control Letters*, 2008, 57, (4), pp.322–331

[15] Peng, C., and Yue, D.: ‘State feedback controller design of networked control systems with parameter uncertainty and state-delay,’ *Asian J. Control*, 2006, 8, (4), pp.385–392

[16] Zhao, H., Wu, M., Liu, G., and She, J.: ‘H-infinity control for networked control systems (NCS) with time-varying delays,’ *J. Control Theory and Applications*, 2005, 3, (2), pp.157–162

[17] Zhu, X., Hua, C., and Wang, S.: ‘State feedback controller design of networked control systems with time delay in the plant,’ *Int. J. Innovative Computing, Information and Control*, 2008, 4, (2), pp.283–290

[18] García-Rivera, M., and Barreiro, A.: ‘Analysis of networked control systems with drops and variable delays,” *Automatica*, 2007, 43, (12), pp.2054–2059

[19] Guan, X., Dai, S., and Long, C.: ‘Controller design for networked control system with data packet dropout and transmission delays,’ *J. Control Theory and Applications*, 2007, 5, (3), pp.227–232

[20] Tian, X., Wang, X., and Cheng, Y.: ‘A self-tuning fuzzy controller for networked control system,’ *Int. J. Computer Science and Network Security*, 2007, 7, (1), pp.97–102

[21] Zhang, W., Branicky, M.S., and Phillips, S.M.: ‘Stability of networked control systems,’ *IEEE Control System Magazine*, 2001, 21, (1), pp.84–99

[22] Gupta, V., Hassibi, B., and Murray, R.M.: ‘Optimal LQG control across packet-dropping links,’ *System and Control Letters*, 2007, 56, (6), pp.439–446

[23] Hu, S., and Yan, W.-Y.: ‘Stability of networked control systems under a multiple-packet transmission policy,’ *IEEE Trans. Automatic Control*, 2008, 53, (7), pp.1706–1711

[24] Zhou, K., Doyle, J.C., and Glover, K.: ‘Robust and Optimal Control’ (Prentice-Hall: Englewood Cliffs, NJ, 1995)

[25] Xu, S., and Chen, T.: 'Robust $\mathcal{H}_\infty$ control for uncertain stochastic systems with state delay,' *IEEE Trans. Automatic Control*, 2002, 47, (12), pp.2089–2094

[26] Yue, D., Han, Q.-L., and Lam, J.: 'Network-based robust H_∞ control of systems with uncertainty,' *Automatica*, 2005, 41, (6), pp.999–1007

[27] Wu, J., Chen, T., and Wang, L.: 'Delay-dependent robust stability and H_∞ control for jump linear systems with delay," *Systems and Control Letters*, 2006, 55, (11), pp.939–948

[28] Wang, Y., and Sun, Z.: 'H_∞ control of networked control system via LMI approach,' *Int. J. Innovative Computing, Information and Control*, 2007, 3, (2), pp.343–352

[29] Yang, D., and Zhang, H.: 'Robust H_∞ networked control for uncertain fuzzy systems with time-delay,' *Acta Automatica Sinica*, 2007, 33, (7), pp.726–730

[30] Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V.: 'Linear Matrix Inequalities in Systems and Control Theory' (SIAM: Philadelphia, PA, 1994)

[31] Wang, Y., Xie, L., and de Souza, C.E.: 'Robust control of a class of uncertain nonlinear systems,' *Systems and Control Letters*, 1992, 19, (2), pp.139–149

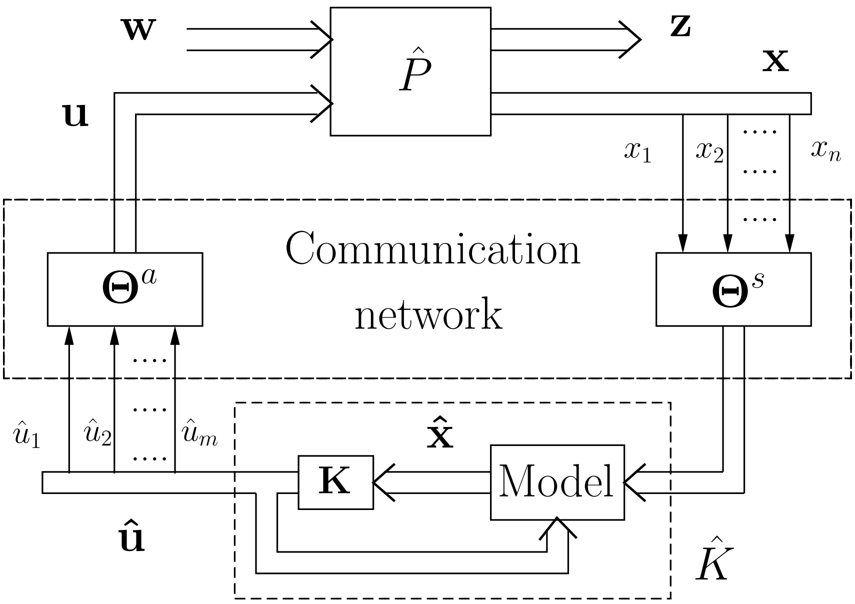

Fig. 1. Networked control system $\hat{P}_K$.

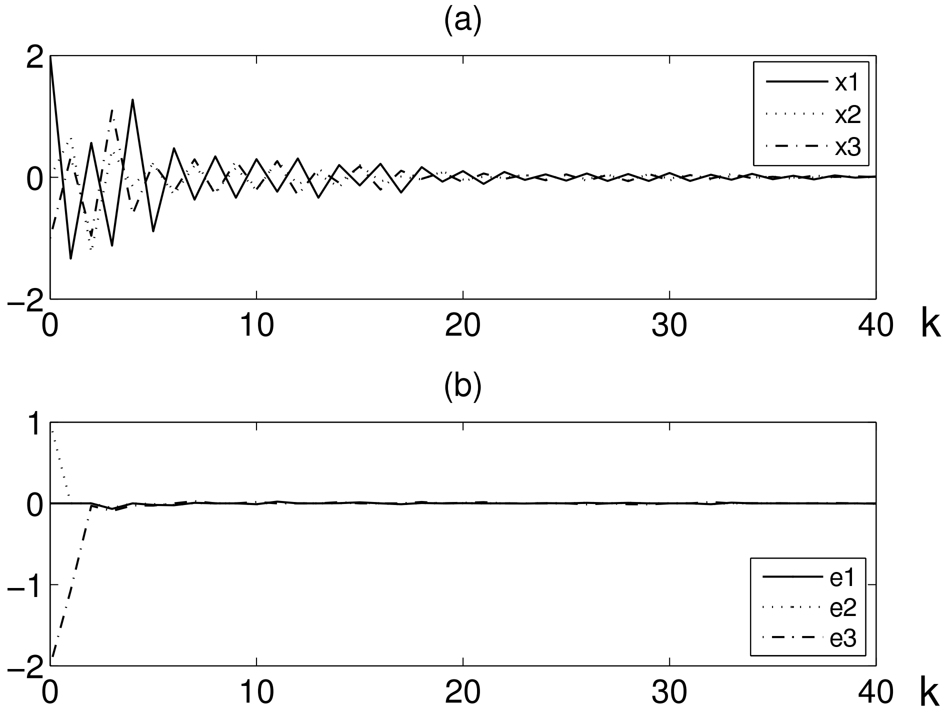

Fig. 2. Typical (a) state trajectories of the plant $\hat{P}$, and (b) error trajectories between the plant and model states.

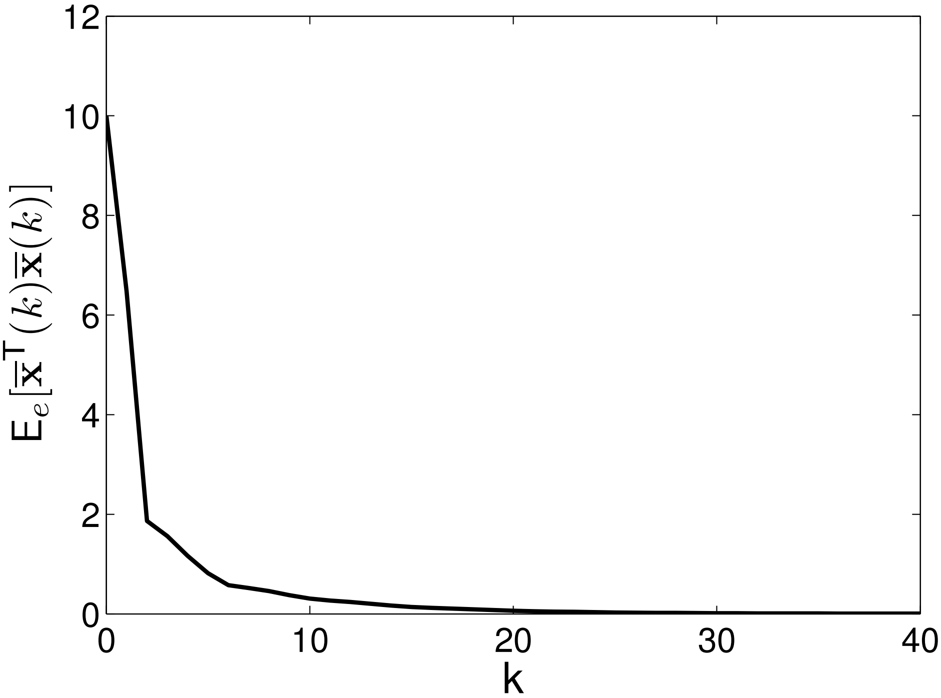

Fig. 3. $\mathsf{E}_e\left[\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)\right]$ calculated by averaging $\overline{\mathbf{x}}^{\mathsf{T}}(k)\overline{\mathbf{x}}(k)$ over 500 simulations.