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Optimal Controller Realisations with the Smallest Dynamic Range

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Outline

- Motivation for optimal **finite word length** controller design with the smallest **dynamic range**

- The proposed two-stage approach for solving this **multi-objective** optimal FWL controller design

- Numerical experimental investigation of the proposed technique

Motivation

- ❑ FWL effect may degrade designed closed-loop performance, and this problem is particularly serious in **fixed-point** implementation

- ❑ Care must be exercised in implementing or **realising** designed control law so as to minimise FWL effect

- ❑ Most existing techniques are based on maximising some FWL **closed-loop stability** measures $\Rightarrow$ far from "optimal":

  - ☆ In fixed-point implementation, total available bits have to accommodate **dynamic range** or integer part, and remaining bits left are then used to implement **precision** or fractional part

  - ☆ Optimising a FWL closed-loop stability measure, while minimising **fractional bit length**, may not guarantee a small dynamic range

Motivation (continue)

- Normalising with l_2-norm will minimise **integer bit length** but may not guarantee adequate FWL closed-loop stability robustness

- True optimal FWL controller design is computationally challenging **multi-objective** optimisation

  - ☆ **Simultaneously** maximise a FWL closed-loop stability measure and minimise a dynamic rage measure

- Our previous work: optimising **combined** FWL closed-loop stability measure and dynamic-range measure

  "A unified closed-loop stability measure for finite-precision digital controller realizations implemented in different representation schemes," IEEE Trans. Automatic Control, 48, pp.816–822, 2003

Proposed Approach

- **True** optimal controller realisation: Simultaneously achieves maximum robustness of FWL closed-loop stability and minimum dynamic range

  We propose a computationally attractive two-step approach to solve this challenging multi-objective optimisation

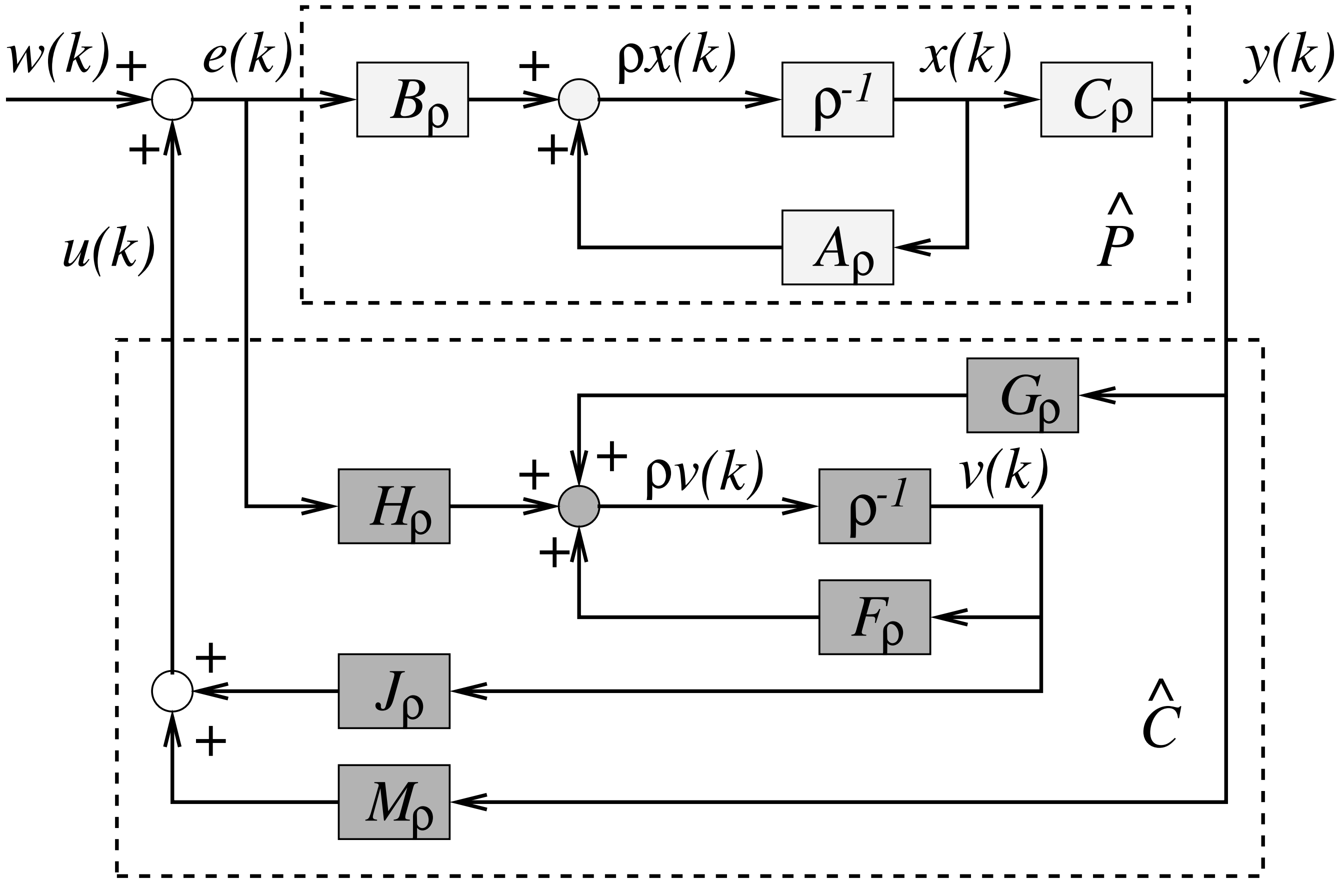
- **Step one**: Maximise FWL closed-loop stability measure
  - ☆ Assuming sufficient integer bit length to avoid overflow, resulting realisation achieves **maximum** robustness of FWL closed-loop stability
  - ☆ We know great deal how to do this
  - ☆ Solution is an infinite set of controller realisations

- **Step two**: Search solution set of optimal FWL closed-loop stability to yield a realisation that has a **minimum** integer bit length

System Model

Discrete-time closed-loop system with **generalised operator** ρ

$$\rho = \begin{cases} z, & \textbf{shift} \\ \delta = \frac{z-1}{h}, & \textbf{delta} \end{cases}$$



System Model (continue)

- State-space description of **plant** $\hat{P}$

$$\begin{cases} \rho\,\mathbf{x}(k) = \mathbf{A}_\rho \mathbf{x}(k) + \mathbf{B}_\rho \mathbf{e}(k) \\ \mathbf{y}(k) = \mathbf{C}_\rho \mathbf{x}(k) \end{cases}$$

$\mathbf{A}_\rho \in \mathcal{R}^{n\times n}$, $\mathbf{B}_\rho \in \mathcal{R}^{n\times p}$ and $\mathbf{C}_\rho \in \mathcal{R}^{q\times n}$

- State-space description of **controller** $\hat{C}$

$$\begin{cases} \rho\,\mathbf{v}(k) = \mathbf{F}_\rho \mathbf{v}(k) + \mathbf{G}_\rho \mathbf{y}(k) + \mathbf{H}_\rho \mathbf{e}(k) \\ \mathbf{u}(k) = \mathbf{J}_\rho \mathbf{v}(k) + \mathbf{M}_\rho \mathbf{y}(k) \end{cases}$$

$\mathbf{F}_\rho \in \mathcal{R}^{m\times m}$, $\mathbf{G}_\rho \in \mathcal{R}^{m\times q}$, $\mathbf{J}_\rho \in \mathcal{R}^{p\times m}$, $\mathbf{M}_\rho \in \mathcal{R}^{p\times q}$ and $\mathbf{H}_\rho \in \mathcal{R}^{m\times p}$

- $\hat{C}$ includes **output feedback**, **full-order observer-based**, and **reduced-order observer-based** controllers

Controller Realisation Set

- Given initial realisation $(\mathbf{F}_{\rho 0}, \mathbf{G}_{\rho 0}, \mathbf{J}_{\rho 0}, \mathbf{M}_{\rho 0}, \mathbf{H}_{\rho 0})$ by standard controller design, all realisations of $\hat{C}$ form **realisation set**

$$\begin{aligned} \mathcal{S}_\rho &= \{(\mathbf{F}_\rho, \mathbf{G}_\rho, \mathbf{J}_\rho, \mathbf{M}_\rho, \mathbf{H}_\rho) : \mathbf{F}_\rho = \mathbf{T}_\rho^{-1}\mathbf{F}_{\rho 0}\mathbf{T}_\rho, \mathbf{G}_\rho = \mathbf{T}_\rho^{-1}\mathbf{G}_{\rho 0}, \\ &\qquad \mathbf{J}_\rho = \mathbf{J}_{\rho 0}\mathbf{T}_\rho, \mathbf{M}_\rho = \mathbf{M}_{\rho 0}, \mathbf{H}_\rho = \mathbf{T}_\rho^{-1}\mathbf{H}_{\rho 0}\} \end{aligned}$$

  $\mathbf{T}_\rho \in \mathcal{R}^{m\times m}$ is any real-valued nonsingular **transformation** matrix

- We can also write a controller realisation in vector form

$$\mathbf{w}_\rho = \left[\mathrm{Vec}^T(\mathbf{F}_\rho)\ \mathrm{Vec}^T(\mathbf{G}_\rho)\ \mathrm{Vec}(\mathbf{J}_\rho)\ \mathrm{Vec}^T(\mathbf{M}_\rho)\ \mathrm{Vec}^T(\mathbf{H}_\rho)\right]^T$$

- **Transition matrix** of closed-loop system

$$\overline{\mathbf{A}}(\mathbf{w}_\rho) = \begin{bmatrix} \mathbf{A}_\rho + \mathbf{B}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{B}_\rho\mathbf{J}_\rho \\ \mathbf{G}_\rho\mathbf{C}_\rho + \mathbf{H}_\rho\mathbf{M}_\rho\mathbf{C}_\rho & \mathbf{F}_\rho + \mathbf{H}_\rho\mathbf{J}_\rho \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_\rho^{-1} \end{bmatrix} \overline{\mathbf{A}}(\mathbf{w}_{\rho 0}) \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_\rho \end{bmatrix}$$

  whose **eigenvalues** are $\lambda_i = \lambda_i(\overline{\mathbf{A}}(\mathbf{w}_\rho))$, $\forall i \in \{1, \cdots, m+n\}$

FWL Robustness

- **Fixed-point** format of bit length $b = 1 + b_g + b_f$: one bit for **sign**, b_g bits for **integer part**, and b_f bits for **fractional part**
- Assume b_g is sufficient so **no overflow** occurs, i.e.

$$\|\mathbf{w}_\rho\|_M \leq 2^{b_g}$$

  where $\|\mathbf{U}\|_M$ denotes maximum absolute element of matrix $\mathbf{U}$
- In FWL implementation, $\mathbf{w}_\rho$ is perturbed into $\mathbf{w}_\rho + \mathbf{\Delta}$ due to **finite** b_f
  - ☆ With **perturbation** $\mathbf{\Delta}$, $\lambda_i(\overline{\mathbf{A}}(\mathbf{w}_\rho))$ moves to $\lambda_i(\overline{\mathbf{A}}(\mathbf{w}_\rho + \mathbf{\Delta}))$
  - ☆ **Will $\overline{\mathbf{A}}(\mathbf{w}_\rho + \mathbf{\Delta})$ remain stable?**
  - ☆ Under condition of no overflow, **closed-loop stability** depends only on $\mathbf{\Delta}$, i.e. **precision** of fractional part representation
- We want a controller **realisation** $\mathbf{w}_\rho$ whose closed-loop stability has maximum **robustness** to controller **perturbation** $\mathbf{\Delta}$

Optimal Realisation

- Optimal FWL **realisation** problem

$$\nu = \min_{\mathbf{w}_\rho \in \mathcal{S}_\rho} f(\mathbf{w}_\rho)$$

- with Frobenius-norm $\| \bullet \|_F$, FWL closed-loop **stability measure**

$$f(\mathbf{w}_\rho) = \max_{i \in \{1,\cdots,m+n\}} \frac{\left\| \frac{\partial \lambda_i}{\partial \mathbf{w}_\rho} \right\|_F}{SM(\lambda_i)}$$

- **Stability margin** of $\lambda_i(\overline{\mathbf{A}}(\mathbf{w}_\rho))$

$$SM(\lambda_i(\overline{\mathbf{A}}(\mathbf{w}_\rho))) = \begin{cases} 1 - |\lambda_i(\overline{\mathbf{A}}(\mathbf{w}_z))|, & \text{if } \rho = z \\ \frac{1}{h} - \left|\lambda_i(\overline{\mathbf{A}}(\mathbf{w}_\delta)) + \frac{1}{h}\right|, & \text{if } \rho = \delta \end{cases}$$

- Note this says nothing about $\|\mathbf{w}_\rho\|_M$ or **dynamic range** of $\mathbf{w}_\rho$

Optimal Realisation Solution

- An **optimal realisation** solution $\mathbf{w}_{\rho\text{opt}}$, i.e. $(\mathbf{F}_{\rho\text{opt}}, \mathbf{G}_{\rho\text{opt}}, \mathbf{J}_{\rho\text{opt}}, \mathbf{M}_{\rho\text{opt}}, \mathbf{H}_{\rho\text{opt}})$, can readily be obtained using algorithm of

  "A search algorithm for a class of optimal finite-precision controller realization problems with saddle points," SIAM J. Control and Optimization, 44, pp.1787–1810, 2005

- This actually defines **optimal solution set** $\mathbf{w}_{\rho\text{opt}}(\mathbf{V})$, where $\mathbf{V} \in \mathcal{R}^{m\times m}$ is an arbitrary **orthogonal matrix**, i.e.

$$\mathcal{S}_{\rho\text{opt}} = \{(\mathbf{F}_\rho, \mathbf{G}_\rho, \mathbf{J}_\rho, \mathbf{M}_\rho, \mathbf{H}_\rho) : \mathbf{F}_\rho = \mathbf{V}^{-1}\mathbf{F}_{\rho\text{opt}}\mathbf{V}, \mathbf{G}_\rho = \mathbf{V}^{-1}\mathbf{G}_{\rho\text{opt}},$$

$$\mathbf{J}_\rho = \mathbf{J}_{\rho\text{opt}}\mathbf{V}, \mathbf{M}_\rho = \mathbf{M}_{\rho\text{opt}}, \mathbf{H}_\rho = \mathbf{V}^{-1}\mathbf{H}_{\rho\text{opt}}, \mathbf{V} \in \mathcal{R}^{m\times m}, \mathbf{V}^T\mathbf{V} = \mathbf{I}\}$$

- Any $\mathbf{w}_{\rho\text{opt}}(\mathbf{V})$ in $\mathcal{S}_{\rho\text{opt}}$ is a solution of optimal FWL realisation problem, but different $\mathbf{w}_{\rho\text{opt}}(\mathbf{V})$ have different **dynamic range** $\|\mathbf{w}_{\rho\text{opt}}(\mathbf{V})\|_M$

Minimising Dynamic Range

- Search $\mathcal{S}_{\rho\text{opt}}$ for a **realisation** with smallest **dynamic range**

$$\mu = \min_{\substack{\mathbf{V} \in \mathcal{R}^{m \times m} \\ \mathbf{V}^T \mathbf{V} = \mathbf{I}}} d(\mathbf{w}_{\rho\text{opt}}(\mathbf{V}))$$

where $d(\mathbf{w}_\rho) = \|\mathbf{w}_\rho\|_M$ is **dynamic range** of $\mathbf{w}_\rho$

- Using Givens rotation with $r = \frac{m(m-1)}{2}$ and $\theta_i \in [-\pi, \pi)$, $1 \le i \le r$

$$d_1(\theta_1, \cdots, \theta_r) = d(\mathbf{w}_{\rho\text{opt}}(\mathbf{V}))$$

- Using optimisation algorithm relying on function value only to solve

$$\mu = \min_{\theta_1, \cdots, \theta_r \in [-\pi, \pi)} d_1(\theta_1, \cdots, \theta_r)$$

With optimal solution $\theta_{1\text{opt}}, \cdots, \theta_{r\text{opt}} \Rightarrow \mathbf{V}_{\text{opt}} \Rightarrow \mathbf{w}_{\rho\text{opt}1} = \mathbf{w}_{\rho\text{opt}}(\mathbf{V}_{\text{opt}})$,
optimal realisation with smallest dynamic range

Numerical Example

- Example from **M. Gevers and G. Li, Parameterizations in Control, Estimation and Filtering Problems: Accuracy Aspects. London: Springer Verlag, 1993**

- **Plant** $\hat{P}$ has order $n = 4$, **controller** $\hat{C}$ is output feedback one with order $m = 4$

- **Initial** controller realisation provided is denoted by $\mathbf{w}_{\rho 0}$

- **Optimal** FWL controller realisation obtained by optimising FWL closed-loop stability measure alone is denoted by $\mathbf{w}_{\rho \mathrm{opt}}$

- Proposed **optimal** FWL controller realisation with **smallest** dynamic range is denoted by $\mathbf{w}_{\rho \mathrm{opt1}}$

Results

- Comparison of three realisations using z operator

Realisation	$f(\mathbf{w}_z)$	$d(\mathbf{w}_z)$	$b_f^{\min}$	$b_g^{\min}$	$b^{\min}$
$\mathbf{w}_{z0}$	$3.9697e+6$	$1.0959e+6$	20	21	42
$\mathbf{w}_{z\mathrm{opt}}$	$2.4246e+3$	$1.9673e+2$	8	8	17
$\mathbf{w}_{z\mathrm{opt1}}$	$2.4246e+3$	$1.1799e+2$	8	7	16

- Comparison of three realisations using δ operator with $h = 2^{-14}$

Realisation	$f(\mathbf{w}_\delta)$	$d(\mathbf{w}_\delta)$	$b_f^{\min}$	$b_g^{\min}$	$b^{\min}$
$\mathbf{w}_{\delta 0}$	$2.7712e+5$	$1.7956e+10$	15	35	51
$\mathbf{w}_{\delta\mathrm{opt}}$	$3.3740e-1$	$5.1236e+4$	-4	16	13
$\mathbf{w}_{\delta\mathrm{opt1}}$	$3.3740e-1$	$2.5810e+4$	-4	15	12

"-4 fractional bits": entire fractional part and first lowest 4-bit integer part are omitted

True Optimal Design

Comparison of $\mathbf{w}_{\delta \mathrm{opt1}}$ under different h

h	$f(\mathbf{w}_{\delta \mathrm{opt1}})$	$d(\mathbf{w}_{\delta \mathrm{opt1}})$	$b_f^{\min}$	$b_g^{\min}$	$b^{\min}$
2^{10}	$2.4825e+6$	$3.6871e+0$	18	2	21
2^{9}	$1.2413e+6$	$5.2144e+0$	17	3	21
2^{8}	$6.2063e+5$	$7.3743e+0$	16	3	20
2^{7}	$3.1032e+5$	$1.0429e+1$	15	4	20
2^{6}	$1.5516e+5$	$1.4749e+1$	14	4	19
2^{5}	$7.7579e+4$	$2.0858e+1$	13	5	19
2^{4}	$3.8790e+4$	$2.9497e+1$	12	5	18
2^{3}	$1.9395e+4$	$4.1715e+1$	11	6	18
2^{2}	$9.6977e+3$	$5.8994e+1$	10	6	17
2^{1}	$4.8490e+3$	$8.3431e+1$	9	7	17
2^{0}	**$2.4246e+3$**	**$1.1799e+2$**	**8**	**7**	**16**
2^{-1}	$1.2125e+3$	$1.6686e+2$	7	8	16
2^{-2}	$6.0639e+2$	$2.3598e+2$	6	8	15
2^{-3}	$3.0335e+2$	$3.3372e+2$	5	9	15
2^{-4}	$1.5183e+2$	$4.7195e+2$	4	9	14
2^{-5}	$7.6071e+1$	$6.6744e+2$	3	10	14
2^{-6}	$3.8190e+1$	$9.4391e+2$	2	10	13
2^{-7}	$1.9248e+1$	$1.3349e+3$	1	11	13
2^{-8}	$9.7758e+0$	$1.8878e+3$	0	11	12
2^{-9}	$5.0361e+0$	$2.6698e+3$	-1	12	12
2^{-10}	**$2.6601e+0$**	**$3.7756e+3$**	**-2**	**12**	**11**
2^{-11}	**$1.4618e+0$**	**$5.3396e+3$**	**-3**	**13**	**11**
2^{-12}	**$8.4740e-1$**	**$7.6314e+3$**	**-3**	**13**	**11**
2^{-13}	$5.2102e-1$	$1.2905e+4$	-3	14	12
2^{-14}	$3.3740e-1$	$2.5810e+4$	-4	15	12
2^{-15}	$2.2681e-1$	$5.1621e+4$	-5	16	12
2^{-16}	$1.5606e-1$	$1.0324e+5$	-6	17	12
2^{-17}	$1.0879e-1$	$2.0648e+5$	-6	18	13
2^{-18}	$7.6367e-2$	$4.1297e+5$	-6	19	14
2^{-19}	$5.3801e-2$	$8.2593e+5$	-7	20	14
2^{-20}	$3.7973e-2$	$1.6519e+6$	-7	21	15
2^{-21}	$2.6826e-2$	$3.3037e+6$	-8	22	15
2^{-22}	$1.8960e-2$	$6.6075e+6$	-8	23	16
2^{-23}	$1.3404e-2$	$1.3215e+7$	-9	24	16
2^{-24}	$9.4767e-3$	$2.6430e+7$	-9	25	17

There exist optimal values of h for the δ operator $\Rightarrow$ resulting optimal controller realisations $\mathbf{w}_{\delta \mathrm{opt1}}$ achieve maximum robustness to FWL errors

Conclusions

- A two-step approach to design optimal **fixed-point** digital controller realisations, which is **multi-objective** optimisation problem

  - ☆ **Step one**: find an optimal realisation by minimising FWL closed-loop stability measure

  - ☆ **Step two**: modifying this realisation to produce optimal realisation with smallest dynamic range

- Approach developed within **unified** framework that includes both shift and delta operator parameterisations of **generic** controller structure

- With appropriate h, optimal δ-operator realisation has much better **FWL closed-loop stability** characteristics than optimal z-operator one

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