Mixed μ Robust Finite Word Length Controller Design

Jun Wu, Gang Li, Sheng Chen and Jian Chu

***Abstract*— A novel finite word length (FWL) controller design is developed in the framework of mixed μ theory. A robust FWL controller performance measure is proposed which takes into account the standard robust control requirements as well as the FWL implementation considerations, and the corresponding FWL robust controller design problem is naturally reformulated as a mixed μ problem which can be treated effectively with the results of mixed μ theory.**

I. Introduction

It is well-known that the detrimental finite word length (FWL) effects cannot be ignored in digital control system designs [1]– [3]. Keel and Bhattacharyya [4] for example showed that digital controllers designed by standard robust control methods may exhibit poor stability margin with respect to the controller coefficient perturbation, if the design does not take into account properly the FWL implementation related uncertainty. There exist mainly two types of FWL errors in digital controller implementation. The first one is the rounding errors that occur in arithmetic operations [5], [6] and the second one is the parameter representation errors [7]– [21], both due to finite precision. Typically, these two types of errors are investigated separately for the reason of mathematical tractability. In this paper we deal with the second type of FWL errors, namely, FWL parameter representation errors.

Two alternative strategies which we refer to as the indirect and direct approaches, respectively, can be used to design digital controllers that take into account FWL parameter representation errors. In the indirect strategy [7]– [13], a control law is firstly constructed by an existing controller synthesis method which may or may not take into account FWL effects. Optimal controller realizations are then selected that are most robust to FWL errors from all the realizations of the given control law. In the direct strategy [14]– [21], the controller realization, which achieves good robust control performance as well as is robust to FWL parameter representation errors, is directly determined in controller synthesis stage by solving the design problem that properly considers FWL effects as well as standard robust control design factors, such as plant uncertainties and disturbances.

This paper adopts the direct approach to consider the robust FWL output feedback controller design. A robust FWL control performance measure is proposed which takes into account the robust control requirements, such as plant uncertainties and input-output characteristics, as well as the FWL effects on controller implementation. We show that the related robust FWL controller design problem can naturally be formulated as a mixed μ problem and thus it can be solved effectively with the aid of the mixed μ theory [22], [23].

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II. Notations and Preliminaries

Let $\mathcal{R}$ be the field of real numbers, $\mathcal{C}$ the field of complex numbers, and $\mathcal{U}$ the closed unit disk in $\mathcal{C}$. $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of matrix $\mathbf{A}$, $\mathbf{A}^*$ the complex conjugate transpose of $\mathbf{A}$, and $\overline{\sigma}(\mathbf{A})$ the largest singular value of $\mathbf{A}$. Let $\rho(\mathbf{A})$ and $\det \mathbf{A}$ represent the spectral radius and the determinant of square matrix $\mathbf{A}$, respectively. $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, while $\mathbf{I}$ and $\mathbf{0}$ represent the identity and zero matrices of appropriate dimensions, respectively. Let $\mathbf{d}_n = [1\ 1 \cdots 1] \in \mathcal{R}^{1\times n}$ whose elements are all equal to 1. $\mathbf{A} \bigotimes \mathbf{B}$ is the Kronecker product of matrices $\mathbf{A}$ and $\mathbf{B}$.

Denote $\mathcal{F}$ the set of all the causal finite-dimensional linear time-invariant discrete-time systems. Any system in $\mathcal{F}$ can be described as

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \end{cases} \quad (1)$$

where $\mathbf{x}(k) \in \mathcal{R}^{n_x}$, $\mathbf{u}(k) \in \mathcal{R}^{n_u}$ and $\mathbf{y}(k) \in \mathcal{R}^{n_y}$ are state, input and output, respectively, and the real constant matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ have appropriate dimensions. The transfer function matrix of the above system is[1]

$$\hat{\mathbf{G}}(\lambda) \stackrel{\triangle}{=} \lambda \mathbf{C}(\mathbf{I} - \lambda\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\,. \quad (2)$$

$\hat{\mathbf{G}}(\lambda)$ is stable ($\mathbf{A}$ is stable) if and only if $\rho(\mathbf{A}) < 1$ or equivalently $\forall \lambda \in \mathcal{U}$, $\det(\mathbf{I} - \lambda\mathbf{A}) \neq 0$. If $\hat{\mathbf{G}}(\lambda)$ is stable, then

$$\|\hat{\mathbf{G}}(\lambda)\|_\infty \stackrel{\triangle}{=} \sup_{\lambda\in\mathcal{U}} \overline{\sigma}(\hat{\mathbf{G}}(\lambda)) < \infty\,.$$

The following results of the mixed μ theory are from [23]. We have a matrix $\mathbf{M} \in \mathcal{C}^{n_a \times n_a}$ and three non-negative integers p, q and r with $p + q + r \leq n_a$, which specify the numbers of uncertainty blocks of three types: repeated

[1] The transfer function matrix is defined as $\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ with the forward shift operator z in most literatures. The backward shift operator $\lambda = z^{-1}$ is adopted in this paper.

complex scalars, repeated real scalars and full complex blocks. A $(p+q+r)$-tuple of positive integers

$$\mathbf{k}(p,q,r) = [k_1 \ \cdots \ k_p \ k_{p+1} \ \cdots \ k_{p+q} \ m_1 \ \cdots \ m_r]^{\mathrm{T}} \quad (3)$$

specifies the dimensions of the perturbation blocks, and we require

$$\sum_{i=1}^{p+q} k_i + \sum_{j=1}^{r} m_j = n_a, \quad (4)$$

in order that these dimensions are compatible with $\mathbf{M}$. The block structure $\mathbf{k}(p,q,r)$ determines the set of allowable perturbations, namely

$$\mathcal{K} \triangleq \left\{ \mathbf{\Upsilon} \left| \begin{array}{l} \mathbf{\Upsilon} = \mathrm{diag}\left(\zeta_1 \mathbf{I}_{k_1}, \cdots, \zeta_p \mathbf{I}_{k_p},\right. \\ \quad \zeta_{p+1}\mathbf{I}_{k_{p+1}}, \cdots, \zeta_{p+q}\mathbf{I}_{k_{p+q}}, \\ \quad \left. \mathbf{\Gamma}_1, \cdots, \mathbf{\Gamma}_r \right) \in \mathcal{C}^{n_a \times n_a} : \\ \forall i \in \{1, \cdots, p\}, \zeta_i \in \mathcal{C}; \\ \forall i \in \{p+1, \cdots, p+q\}, \zeta_i \in \mathcal{R}; \\ \forall j \in \{1, \cdots, r\}, \mathbf{\Gamma}_j \in \mathcal{C}^{m_j \times m_j} \end{array} \right. \right\}. \quad (5)$$

The mixed μ of a matrix $\mathbf{M} \in \mathcal{C}^{n_a \times n_a}$ with respect to a perturbation set $\mathcal{K}$ is defined as

$$\mu_{\mathcal{K}}(\mathbf{M}) \triangleq \left(\inf_{\mathbf{\Upsilon} \in \mathcal{K}} \{ \overline{\sigma}(\mathbf{\Upsilon}) | \det(\mathbf{I} - \mathbf{\Upsilon}\mathbf{M}) = 0 \} \right)^{-1}. \quad (6)$$

with $\mu_{\mathcal{K}}(\mathbf{M}) = 0$ if no $\mathbf{\Upsilon} \in \mathcal{K}$ solves $\det(\mathbf{I} - \mathbf{\Upsilon}\mathbf{M}) = 0$.

Presently, except for a few special cases, how to compute $\mu_{\mathcal{K}}(\mathbf{M})$ is unknown. However, an upper bound of $\mu_{\mathcal{K}}(\mathbf{M})$ provided in the following is easy to compute and is often used to replace $\mu_{\mathcal{K}}(\mathbf{M})$ in practice. Define

$$\mathcal{E}_{\mathcal{K}} \triangleq \left\{ \mathbf{E} \left| \begin{array}{l} \mathbf{E} = \mathrm{diag}\left(\mathbf{E}_1, \cdots, \mathbf{E}_p, \right. \\ \quad \mathbf{E}_{p+1}, \cdots, \mathbf{E}_{p+q}, \\ \quad \left. \eta_1 \mathbf{I}_{m_1}, \cdots, \eta_r \mathbf{I}_{m_r} \right) \in \mathcal{C}^{n_a \times n_a} : \\ \forall i \in \{1, \cdots, p+q\}, \\ \quad 0 < \mathbf{E}_i \in \mathcal{C}^{k_i \times k_i}; \\ \forall j \in \{1, \cdots, r\}, 0 < \eta_j \in \mathcal{R} \end{array} \right. \right\}, \quad (7)$$

$$\mathcal{G}_{\mathcal{K}} \triangleq \left\{ \mathbf{G} \left| \begin{array}{l} \mathbf{G} = \mathrm{diag}\left(0\mathbf{I}_{k_1}, \cdots, 0\mathbf{I}_{k_p}, \right. \\ \quad \mathbf{G}_{p+1}, \cdots, \mathbf{G}_{p+q}, \\ \quad \left. 0\mathbf{I}_{m_1}, \cdots, 0\mathbf{I}_{m_r} \right) \in \mathcal{C}^{n_a \times n_a} : \\ \forall i \in \{p+1, \cdots, p+q\}, \\ \quad \mathbf{G}_i = \mathbf{G}_i^* \in \mathcal{C}^{k_i \times k_i} \end{array} \right. \right\}. \quad (8)$$

Then

$$\alpha_{\mathcal{K}}(\mathbf{M}) \triangleq \inf_{\substack{\mathbf{E} \in \mathcal{E}_{\mathcal{K}} \\ \mathbf{G} \in \mathcal{G}_{\mathcal{K}} \\ 0 < \alpha \in \mathcal{R}}} \left\{ \alpha \left| \begin{array}{l} \alpha^2 \mathbf{E} - \mathbf{M}^* \mathbf{E} \mathbf{M} \\ -\sqrt{-1}(\mathbf{G}\mathbf{M} - \mathbf{M}^*\mathbf{G}) > 0 \end{array} \right. \right\} \quad (9)$$

is an upper bound of $\mu_{\mathcal{K}}(\mathbf{M})$, i.e. $\mu_{\mathcal{K}}(\mathbf{M}) \leq \alpha_{\mathcal{K}}(\mathbf{M})$. When the real scalars of $\mathbf{\Upsilon} \in \mathcal{K}$ are not repeated and $\mathbf{M}$ is a real matrix, $\alpha_{\mathcal{K}}(\mathbf{M})$ can be expressed in a simpler form and computed more easily. Define

$$\mathcal{E}_{\mathcal{RK}} \triangleq \left\{ \mathbf{E} \in \mathcal{E}_{\mathcal{K}} \left| \mathbf{E} \in \mathcal{R}^{n_a \times n_a} \right. \right\}. \quad (10)$$

The following lemma is Theorem 5.12 in [23].

Lemma 1: Let a real matrix $\mathbf{M} \in \mathcal{R}^{n_a \times n_a}$ and a perturbation set $\mathcal{K}$ with $k_i = 1$ for $i \in \{p+1, \cdots, p+q\}$ (i.e. none of the real scalars are repeated). Then

$$\alpha_{\mathcal{K}}(\mathbf{M}) = \inf_{\substack{\mathbf{E} \in \mathcal{E}_{\mathcal{RK}} \\ 0 < \alpha \in \mathcal{R}}} \{ \alpha \mid \alpha^2 \mathbf{E} - \mathbf{M}^{\mathrm{T}} \mathbf{E} \mathbf{M} > 0 \}. \quad (11)$$

Corollary 1: For $\mathbf{M}$ and $\mathcal{K}$ as in *Lemma 1*, $\alpha_{\mathcal{K}}(\mathbf{M}) < 1$ if and only if there exists $\mathbf{E} \in \mathcal{E}_{\mathcal{RK}}$ such that $\mathbf{E} - \mathbf{M}^{\mathrm{T}}\mathbf{E}\mathbf{M} > 0$.

III. Robust FWL Performance Measure

The plant is described by a known nominal model $\hat{\mathbf{P}}_g(\lambda)$ and an uncertainty $\hat{\mathbf{H}}(\lambda)$ which is unknown but bounded. $\hat{\mathbf{P}}_g(\lambda)$ is given as

$$\left\{ \begin{array}{rcl} \mathbf{x}_P(k+1) & = & \mathbf{A}_P \mathbf{x}_P(k) + \mathbf{B}_v \mathbf{v}(k) + \\ & & \mathbf{B}_w \mathbf{w}(k) + \mathbf{B}_P \mathbf{u}_P(k), \\ \mathbf{h}(k) & = & \mathbf{C}_h \mathbf{x}_P(k) + \\ & & \mathbf{D}_{1,1}\mathbf{v}(k) + \mathbf{D}_{1,2}\mathbf{w}(k), \\ \mathbf{z}(k) & = & \mathbf{C}_z \mathbf{x}_P(k) + \\ & & \mathbf{D}_{2,1}\mathbf{v}(k) + \mathbf{D}_{2,2}\mathbf{w}(k), \\ \mathbf{y}_P(k) & = & \mathbf{C}_P \mathbf{x}_P(k), \end{array} \right. \quad (12)$$

where $\mathbf{x}_P(k) \in \mathcal{R}^n$, $\mathbf{v}(k) \in \mathcal{R}^{n_1}$, $\mathbf{w}(k) \in \mathcal{R}^{n_2}$, $\mathbf{u}_P(k) \in \mathcal{R}^s$, $\mathbf{h}(k) \in \mathcal{R}^{n_1}$, $\mathbf{z}(k) \in \mathcal{R}^{n_2}$, $\mathbf{y}_P(k) \in \mathcal{R}^t$, $\mathbf{w}(k)$ is the external disturbance input, and $\mathbf{z}(k)$ is the controlled output. We have assumed without loss of generality that $\mathbf{v}(k)$ and $\mathbf{h}(k)$ have the same dimension while $\mathbf{w}(k)$ and $\mathbf{z}(k)$ have the same dimension. If the paired variables have different dimensions, they can be made equal by adding an appropriate number of zero rows/columns to the corresponding plant matrices. In addition, it is assumed that $\mathbf{B}_P^{\mathrm{T}}\mathbf{B}_P > 0$ and $\mathbf{C}_P\mathbf{C}_P^{\mathrm{T}} > 0$. This assumption reflects a reasonable practical situation of no redundant actuator or sensor. Through $\mathbf{h}(k)$ and $\mathbf{v}(k)$, $\hat{\mathbf{P}}_g(\lambda)$ connects with $\hat{\mathbf{H}}(\lambda)$, i.e.

$$\mathbf{v} = \hat{\mathbf{H}}(\lambda)\mathbf{h}. \quad (13)$$

$\hat{\mathbf{H}}(\lambda)$ is included in the set

$$\mathcal{H}_\tau \triangleq \left\{ \hat{\mathbf{H}}(\lambda) \left| \begin{array}{l} \hat{\mathbf{H}}(\lambda) \in \mathcal{F}, \hat{\mathbf{H}}(\lambda) \text{ is stable,} \\ \|\hat{\mathbf{H}}(\lambda)\|_\infty < \tau \end{array} \right. \right\} \quad (14)$$

with a given constant $\tau > 0$. The digital controller $\hat{\mathbf{C}}(\lambda)$ of mth-order is described by

$$\left\{ \begin{array}{rcl} \mathbf{x}_C(k+1) & = & \mathbf{A}_C \mathbf{x}_C(k) + \mathbf{B}_C \mathbf{y}_P(k) \\ \mathbf{u}_P(k) & = & \mathbf{C}_C \mathbf{x}_C(k) + \mathbf{D}_C \mathbf{y}_P(k) \end{array} \right. \quad (15)$$

with $\mathbf{A}_C \in \mathcal{R}^{m \times m}$, $\mathbf{B}_C \in \mathcal{R}^{m \times t}$, $\mathbf{C}_C \in \mathcal{R}^{s \times m}$ and $\mathbf{D}_C \in \mathcal{R}^{s \times t}$. Let us denote

$$\mathbf{X} \triangleq \left[\begin{array}{cc} \mathbf{D}_C & \mathbf{C}_C \\ \mathbf{B}_C & \mathbf{A}_C \end{array} \right] \in \mathcal{R}^{(s+m) \times (t+m)}. \quad (16)$$

Denote furthermore

$$N \triangleq (s+m)(t+m), \quad (17)$$

$$\mathcal{O} \triangleq \{ \mathbf{\Lambda} \mid \mathbf{\Lambda} \in \mathcal{R}^{N \times N}, \mathbf{\Lambda} \text{ is diagonal} \}, \quad (18)$$

$$\mathcal{O}_\beta \triangleq \{ \mathbf{\Lambda} \mid \mathbf{\Lambda} \in \mathcal{O}, \overline{\sigma}(\mathbf{\Lambda}) \leq \beta \}. \quad (19)$$

When $\mathbf{X}$ is implemented in fixed-point format of FWL, it is perturbed into $\mathbf{X} + (\mathbf{d}_{t+m} \bigotimes \mathbf{I}_{s+m})\mathbf{\Lambda}(\mathbf{I}_{t+m} \bigotimes \mathbf{d}_{s+m}^{\mathrm{T}})$, where $\mathbf{\Lambda} \in \mathcal{O}_\beta$ and $0 \leq \beta \in \mathcal{R}$ is the maximum representation error of the fixed-point digital processor.

The above description represents a closed-loop system consisting of $\hat{\mathbf{P}}_g(\lambda)$ and $\hat{\mathbf{H}}(\lambda)$ as well as $\mathbf{X}$ and $\mathbf{\Lambda}$. Denote this closed-loop system as $\hat{\mathbf{\Phi}}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{\Lambda})$ and the closed-loop transfer function from $\mathbf{w}(k)$ to $\mathbf{z}(k)$ as $\hat{\mathbf{\Phi}}_{wz}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{\Lambda})$. For $0 < \xi \in \mathcal{R}$, a set is defined which consists of all the mth-order robust controllers without FWL consideration, that is,

$$\mathcal{X}_m \triangleq \left\{ \mathbf{X} \left| \begin{array}{c} \mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}, \forall \hat{\mathbf{H}}(\lambda) \in \mathcal{H}_\tau, \\ \hat{\mathbf{\Phi}}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{0}) \text{ is stable}, \\ \|\hat{\mathbf{\Phi}}_{wz}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{0})\|_\infty \leq \xi \end{array} \right. \right\}. \quad (20)$$

To take into account the FWL error $\mathbf{\Lambda}$, we propose the following FWL performance measure for $\mathbf{X} \in \mathcal{X}_m$

$$\upsilon_d(\mathbf{X}) \triangleq \sup_{0 \leq \beta \in \mathcal{R}} \left\{ \beta \left| \begin{array}{l} \forall \hat{\mathbf{H}}(\lambda) \in \mathcal{H}_\tau, \forall \mathbf{\Lambda} \in \mathcal{O}_\beta, \\ \hat{\mathbf{\Phi}}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{\Lambda}) \text{ is stable}, \\ \|\hat{\mathbf{\Phi}}_{wz}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{\Lambda})\|_\infty \leq \xi \end{array} \right. \right\}. \quad (21)$$

For a given $\mathbf{X} \in \mathcal{X}_m$, how to compute the value of $\upsilon_d(\mathbf{X})$ is unknown. Therefore, a tractable lower bound of $\upsilon_d(\mathbf{X})$ is derived with the aid of mixed μ. By "pulling out" $\hat{\mathbf{H}}(\lambda)$ and considering the composite system of $\hat{\mathbf{P}}_g(\lambda)$, $\mathbf{X}$ and $\mathbf{\Lambda}$, the description of this composite system can be obtained as

$$\left\{ \begin{array}{rcl} \mathbf{x}_{PC}(k+1) & = & \left(\overline{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u \mathbf{\Lambda} \mathbf{C}_u\right) \mathbf{x}_{PC}(k) \\ & & + \mathbf{B}_{\overline{v}} \mathbf{v}(k) + \mathbf{B}_{\overline{w}} \mathbf{w}(k), \\ \mathbf{h}(k) & = & \mathbf{C}_{\overline{h}} \mathbf{x}_{PC}(k) + \mathbf{D}_{1,1} \mathbf{v}(k) \\ & & + \mathbf{D}_{1,2} \mathbf{w}(k), \\ \mathbf{z}(k) & = & \mathbf{C}_{\overline{z}} \mathbf{x}_{PC}(k) + \mathbf{D}_{2,1} \mathbf{v}(k) \\ & & + \mathbf{D}_{2,2} \mathbf{w}(k), \end{array} \right. \quad (22)$$

where

$$\begin{aligned} \overline{\mathbf{A}}(\mathbf{X}) &= \left[\begin{array}{cc} \mathbf{A}_P + \mathbf{B}_P \mathbf{D}_C \mathbf{C}_P & \mathbf{B}_P \mathbf{C}_C \\ \mathbf{B}_C \mathbf{C}_P & \mathbf{A}_C \end{array} \right] \\ &= \left[\begin{array}{cc} \mathbf{A}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{array} \right] + \left[\begin{array}{cc} \mathbf{B}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{array} \right] \mathbf{X} \left[\begin{array}{cc} \mathbf{C}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{array} \right] \\ &\triangleq \mathbf{M}_0 + \mathbf{M}_1 \mathbf{X} \mathbf{M}_2 \in \mathcal{R}^{(n+m)\times(n+m)}. \end{aligned} \quad (23)$$

$$\mathbf{B}_u \triangleq \mathbf{d}_{t+m} \bigotimes \mathbf{M}_1 \in \mathcal{R}^{(n+m)\times N}, \quad (24)$$

$$\mathbf{C}_u \triangleq \mathbf{M}_2 \bigotimes \mathbf{d}_{s+m}^{\mathrm{T}} \in \mathcal{R}^{N\times(n+m)}, \quad (25)$$

$$\mathbf{B}_{\overline{v}} = \left[\begin{array}{c} \mathbf{B}_v \\ \mathbf{0} \end{array} \right] \in \mathcal{R}^{(n+m)\times n_1}, \quad (26)$$

$$\mathbf{B}_{\overline{w}} = \left[\begin{array}{c} \mathbf{B}_w \\ \mathbf{0} \end{array} \right] \in \mathcal{R}^{(n+m)\times n_2}, \quad (27)$$

$$\mathbf{C}_{\overline{h}} = \left[\begin{array}{cc} \mathbf{C}_h & \mathbf{0} \end{array} \right] \in \mathcal{R}^{n_1\times(n+m)}, \quad (28)$$

$$\mathbf{C}_{\overline{z}} = \left[\begin{array}{cc} \mathbf{C}_z & \mathbf{0} \end{array} \right] \in \mathcal{R}^{n_2\times(n+m)}, \quad (29)$$

and

$$\mathbf{x}_{PC}(k) = \left[\begin{array}{c} \mathbf{x}_P(k) \\ \mathbf{x}_C(k) \end{array} \right].$$

When the system (22) is stable, its transfer function matrix is defined by

$$\begin{aligned} \hat{\mathbf{\Psi}}(\lambda, \mathbf{X}, \mathbf{\Lambda}) \triangleq\ & \lambda \left[\begin{array}{c} \mathbf{C}_{\overline{h}} \\ \mathbf{C}_{\overline{z}} \end{array} \right] (\mathbf{I} - \lambda(\overline{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u \mathbf{\Lambda} \mathbf{C}_u))^{-1} \\ & \times \left[\begin{array}{cc} \mathbf{B}_{\overline{v}} & \mathbf{B}_{\overline{w}} \end{array} \right] + \left[\begin{array}{cc} \mathbf{D}_{1,1} & \mathbf{D}_{1,2} \\ \mathbf{D}_{2,1} & \mathbf{D}_{2,2} \end{array} \right] \end{aligned} \quad (30)$$

where $\hat{\mathbf{\Psi}}(\lambda, \mathbf{X}, \mathbf{\Lambda}) \in \mathcal{C}^{(n_1+n_2)\times(n_1+n_2)}$. For all $\lambda \in \mathcal{U}$, let

$$\mathcal{K}_\psi \triangleq \left\{ \left[\begin{array}{cc} \mathbf{\Upsilon}_{\psi 1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Upsilon}_{\psi 2} \end{array} \right] \left| \begin{array}{c} \mathbf{\Upsilon}_{\psi 1} \in \mathcal{C}^{n_1\times n_1}, \\ \mathbf{\Upsilon}_{\psi 2} \in \mathcal{C}^{n_2\times n_2} \end{array} \right. \right\}. \quad (31)$$

Accordingly, we can obtain $\mu_{\mathcal{K}_\psi}(\hat{\mathbf{\Psi}}(\lambda, \mathbf{X}, \mathbf{\Lambda}))$ for all $\lambda \in \mathcal{U}$. The following result on robust performance [24] links $\upsilon_d(\mathbf{X})$ to mixed μ.

Lemma 2: For $\mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}$, if and only if there exists $0 \leq \beta \in \mathcal{R}$ such that

$$\hat{\mathbf{\Psi}}(\lambda, \mathbf{X}, \mathbf{\Lambda}) \text{ is stable}, \ \forall \mathbf{\Lambda} \in \mathcal{O}_\beta, \quad (32)$$

$$\left\{ \begin{array}{l} \mu_{\mathcal{K}_\psi}\left(\left[\begin{array}{cc} \tau \mathbf{I}_{n_1} & \\ & \frac{1}{\xi} \mathbf{I}_{n_2} \end{array} \right] \hat{\mathbf{\Psi}}(\lambda, \mathbf{X}, \mathbf{\Lambda}) \right) \leq 1, \\ \qquad \forall \lambda \in \mathcal{U}, \ \forall \mathbf{\Lambda} \in \mathcal{O}_\beta, \end{array} \right. \quad (33)$$

then $\mathbf{X} \in \mathcal{X}_m$ and $\forall \hat{\mathbf{H}}(\lambda) \in \mathcal{H}_\tau$, $\forall \mathbf{\Lambda} \in \mathcal{O}_\beta$, $\hat{\mathbf{\Phi}}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{\Lambda})$ is stable, $\|\hat{\mathbf{\Phi}}_{wz}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{\Lambda})\|_\infty \leq \xi$.

Clearly, by replacing (33) with

$$\left\{ \begin{array}{l} \mu_{\mathcal{K}_\psi}\left(\left[\begin{array}{cc} \tau \mathbf{I}_{n_1} & \\ & \frac{1}{\xi} \mathbf{I}_{n_2} \end{array} \right] \hat{\mathbf{\Psi}}(\lambda, \mathbf{X}, \mathbf{\Lambda}) \right) < 1, \\ \qquad \forall \lambda \in \mathcal{U}, \ \forall \mathbf{\Lambda} \in \mathcal{O}_\beta, \end{array} \right. \quad (34)$$

we have a sufficient and "almost necessary" condition. The problem in dealing with (32) and (34) is that $\hat{\mathbf{\Psi}}(\lambda, \mathbf{X}, \mathbf{\Lambda})$ contains indeterminate λ and $\mathbf{\Lambda}$. For this reason, we first transform (32) and (34).

Theorem 1: For $\mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}$, if and only if there exists $0 \leq \beta \in \mathcal{R}$ such that

$$\mu_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X}, \beta)) < 1, \quad (35)$$

then (32) and (34) hold. In (35), $\mathbf{\Theta}(\mathbf{X}, \beta)$ and its corresponding perturbation set $\mathcal{K}_\theta$ are defined respectively as

$$\mathbf{\Theta}(\mathbf{X}, \beta) \triangleq \left[\begin{array}{cccc} \overline{\mathbf{A}}(\mathbf{X}) & \mathbf{B}_u & \mathbf{B}_{\overline{v}} & \mathbf{B}_{\overline{w}} \\ \beta \mathbf{C}_u & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \tau \mathbf{C}_{\overline{h}} & \mathbf{0} & \tau \mathbf{D}_{1,1} & \tau \mathbf{D}_{1,2} \\ \frac{1}{\xi} \mathbf{C}_{\overline{z}} & \mathbf{0} & \frac{1}{\xi} \mathbf{D}_{2,1} & \frac{1}{\xi} \mathbf{D}_{2,2} \end{array} \right], \quad (36)$$

where $\mathbf{\Theta}(\mathbf{X}, \beta) \in \mathcal{R}^{(n+m+N+n_1+n_2)\times(n+m+N+n_1+n_2)}$, and

$$\mathcal{K}_\theta \triangleq \left\{ \left[\begin{array}{cc} \mathbf{\Upsilon}_h & \\ & \mathbf{\Upsilon}_\psi \end{array} \right] \left| \begin{array}{c} \mathbf{\Upsilon}_h \in \mathcal{K}_h, \\ \mathbf{\Upsilon}_\psi \in \mathcal{K}_\psi \end{array} \right. \right\}. \quad (37)$$

Due to the well-known difficulty in computing the value of $\mu_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X}, \beta))$, we replace $\mu_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X}, \beta))$ with $\alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X}, \beta))$.

Corollary 2: For $\mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}$, if there exists $0 \leq \beta \in \mathcal{R}$ such that

$$\alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X}, \beta)) < 1, \quad (38)$$

then $\mathbf{X} \in \mathcal{X}_m$ and $\forall \hat{\mathbf{H}}(\lambda) \in \mathcal{H}_\tau$, $\forall \mathbf{\Lambda} \in \mathcal{O}_\beta$, $\hat{\mathbf{\Phi}}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{\Lambda})$ is stable, $\|\hat{\mathbf{\Phi}}_{wz}(\lambda, \hat{\mathbf{H}}(\lambda), \mathbf{X}, \mathbf{\Lambda})\|_\infty \leq \xi$.

Because $\alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},\beta)) \geq \mu_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},\beta))$, (38) is a sufficient condition for (35) to hold. Based on Corollary 2, define

$$\widetilde{\mathcal{X}}_m \triangleq \{\mathbf{X} \mid \mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}, \alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},0)) < 1\}, \quad (39)$$

which obviously is a subset of $\mathcal{X}_m$. For $\mathbf{X} \in \widetilde{\mathcal{X}}_m$, define

$$\widetilde{\upsilon}_d(\mathbf{X}) \triangleq \sup_{0\leq\beta\in\mathcal{R}} \{\beta \mid \alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},\beta)) < 1\}, \quad (40)$$

which obviously is a lower bound of $\upsilon_d(\mathbf{X})$ and is an FWL performance measure. For $\mathcal{K}_\theta$ given in (37), the related positive definite matrix set

$$\mathcal{E}_{\mathcal{R}\mathcal{K}_\theta} \triangleq \left\{ \mathbf{E} \;\middle|\; \begin{array}{l} \mathbf{E} = \mathrm{diag}\,(\mathbf{E}_1, e_1, \cdots, e_N, \\ \qquad \eta_1 \mathbf{I}_{n_1}, \eta_2 \mathbf{I}_{n_2}), \\ 0 < \mathbf{E}_1 \in \mathcal{R}^{(n+m)\times(n+m)}, \\ 0 < e_1, \cdots, e_N, \eta_1, \eta_2 \in \mathcal{R} \end{array} \right\} \quad (41)$$

is defined. It is interesting to see that $\mathbf{\Theta}(\mathbf{X},\beta)$ and $\mathcal{K}_\theta$ satisfy the condition of Corollary 1 and hence $\widetilde{\upsilon}_d(\mathbf{X})$ is computable by solving the following optimisation problem

$$\begin{aligned} \widetilde{\upsilon}_d(\mathbf{X}) &= \sup_{0\leq\beta\in\mathcal{R}} \beta, & (42)\\ \text{s.t.} \quad & \mathbf{E} > \mathbf{\Theta}^{\mathrm{T}}(\mathbf{X},\beta)\mathbf{E}\mathbf{\Theta}(\mathbf{X},\beta), \\ & \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta}, \end{aligned}$$

based on the combined linear matrix inequality (LMI) technique [25] and bisection search [26].

IV. Robust FWL Controller Design

With the tractable FWL performance measure $\widetilde{\upsilon}_d(\mathbf{X})$, the proposed FWL controller design problem can now be summarised. Given $\hat{\mathbf{P}}_g(\lambda)$, τ, ξ, m and assuming a nonempty $\widetilde{\mathcal{X}}_m$, find a controller realization $\mathbf{X} \in \widetilde{\mathcal{X}}_m$ that achieves

$$\gamma_d = \sup_{\mathbf{X}\in\widetilde{\mathcal{X}}_m} \widetilde{\upsilon}_d(\mathbf{X}), \quad (43)$$

or equivalently

$$\begin{aligned} \gamma_d &= \sup_{0\leq\beta\in\mathcal{R}} \beta, & (44)\\ \text{s.t.} \quad & \mathbf{E} > \mathbf{\Theta}^{\mathrm{T}}(\mathbf{X},\beta)\mathbf{E}\mathbf{\Theta}(\mathbf{X},\beta), \\ & \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta},\ \mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}. \end{aligned}$$

Note that this optimisation problem contains a bilinear matrix inequality (BMI) [27] of size $2(n+m+N+n_1+n_2)$. Since

$$\mathbf{\Theta}(\mathbf{X},\beta) = \mathbf{Y}_\beta + \mathbf{Y}_1\mathbf{X}\mathbf{Y}_2 \quad (45)$$

with

$$\mathbf{Y}_\beta \triangleq \begin{bmatrix} \mathbf{M}_0 & \mathbf{B}_u & \mathbf{B}_{\overline{v}} & \mathbf{B}_{\overline{w}} \\ \beta\mathbf{C}_u & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \tau\mathbf{C}_{\overline{h}} & \mathbf{0} & \tau\mathbf{D}_{1,1} & \tau\mathbf{D}_{1,2} \\ \frac{1}{\xi}\mathbf{C}_{\overline{z}} & \mathbf{0} & \frac{1}{\xi}\mathbf{D}_{2,1} & \frac{1}{\xi}\mathbf{D}_{2,2} \end{bmatrix}, \quad (46)$$

where $\mathbf{Y}_\beta \in \mathcal{R}^{(n+m+N+n_1+n_2)\times(n+m+N+n_1+n_2)}$, and

$$\mathbf{Y}_1 \triangleq \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{0} \end{bmatrix} \in \mathcal{R}^{(n+m+N+n_1+n_2)\times(s+m)}, \quad (47)$$

$$\mathbf{Y}_2 \triangleq \begin{bmatrix} \mathbf{M}_2 & \mathbf{0} \end{bmatrix} \in \mathcal{R}^{(t+m)\times(n+m+N+n_1+n_2)}, \quad (48)$$

we can write the optimisation problem (44) as

$$\begin{aligned} \gamma_d &= \sup_{0\leq\beta\in\mathcal{R}} \beta, & (49)\\ \text{s.t.} \quad & \mathbf{E} > (\mathbf{Y}_\beta + \mathbf{Y}_1\mathbf{X}\mathbf{Y}_2)^{\mathrm{T}}\mathbf{E}(\mathbf{Y}_\beta + \mathbf{Y}_1\mathbf{X}\mathbf{Y}_2), \\ & \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta},\ \mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}. \end{aligned}$$

The following result [28], [29] is useful in solving the optimisation problem (49).

Lemma 3: Suppose that $\mathbf{Y}_1^{\mathrm{T}}\mathbf{Y}_1 > 0$ and $\mathbf{Y}_2\mathbf{Y}_2^{\mathrm{T}} > 0$. Give a $0 < \omega \in \mathcal{R}$ and a $0 \leq \beta \in \mathcal{R}$. If and only if there exist $0 < \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta}$, $\mathbf{J} \in \mathcal{R}^{(s+m)\times(n+m+N+n_1+n_2)}$ and $\mathbf{L} \in \mathcal{R}^{(n+m+N+n_1+n_2)\times(t+m)}$ such that

$$\left\{ \begin{array}{l} \omega\mathbf{E} > (\mathbf{Y}_\beta + \mathbf{Y}_1\mathbf{J})^{\mathrm{T}}\mathbf{E}(\mathbf{Y}_\beta + \mathbf{Y}_1\mathbf{J}), \\ \omega\mathbf{E} > (\mathbf{Y}_\beta + \mathbf{L}\mathbf{Y}_2)^{\mathrm{T}}\mathbf{E}(\mathbf{Y}_\beta + \mathbf{L}\mathbf{Y}_2), \end{array} \right. \quad (50)$$

then there exists $\mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}$ such that

$$\omega\mathbf{E} > (\mathbf{Y}_\beta + \mathbf{Y}_1\mathbf{X}\mathbf{Y}_2)^{\mathrm{T}}\mathbf{E}(\mathbf{Y}_\beta + \mathbf{Y}_1\mathbf{X}\mathbf{Y}_2). \quad (51)$$

When (50) holds, all the $\mathbf{X}$ satisfying (51) can be expressed as

$$\begin{aligned} \mathbf{X} &= -(\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}\mathbf{Y}_1)^{-1}\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}\mathbf{Y}_\beta\mathbf{\Xi}_1\mathbf{Y}_2^{\mathrm{T}}(\mathbf{Y}_2\mathbf{\Xi}_1\mathbf{Y}_2^{\mathrm{T}})^{-1} \\ &\quad +\mathbf{\Xi}_2^{1/2}\mathbf{\Omega}(\mathbf{Y}_2\mathbf{\Xi}_1\mathbf{Y}_2^{\mathrm{T}})^{-1} \qquad (52) \end{aligned}$$

where

$$\mathbf{\Xi}_1 \triangleq (\omega\mathbf{E} - \mathbf{Y}_\beta^{\mathrm{T}}\mathbf{E}\mathbf{Y}_\beta + \mathbf{Y}_\beta^{\mathrm{T}}\mathbf{E}\mathbf{Y}_1(\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}\mathbf{Y}_1)^{-1}\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}\mathbf{Y}_\beta)^{-1}, \quad (53)$$

$$\begin{aligned} \mathbf{\Xi}_2 &\triangleq (\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}\mathbf{Y}_1)^{-1} - (\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}\mathbf{Y}_1)^{-1}\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}\mathbf{Y}_\beta \\ &\quad \times(\mathbf{\Xi}_1 - \mathbf{\Xi}_1\mathbf{Y}_2^{\mathrm{T}}(\mathbf{Y}_2\mathbf{\Xi}_1\mathbf{Y}_2^{\mathrm{T}})^{-1}\mathbf{Y}_2\mathbf{\Xi}_1) \\ &\quad \times\mathbf{Y}_\beta^{\mathrm{T}}\mathbf{E}\mathbf{Y}_1(\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}\mathbf{Y}_1)^{-1}, & (54)\\ \mathbf{\Omega} &\in \mathcal{R}^{(s+m)\times(t+m)},\ \overline{\sigma}(\mathbf{\Omega}) < 1. & (55) \end{aligned}$$

The above lemma shows that (51) can be transformed into (50). It is easy to see that (50) actually is an LMI when $\mathbf{J}$ is given. Moreover, (50) is equivalent to

$$\left\{ \begin{array}{l} \omega\mathbf{E}^{-1} > (\mathbf{Y}_\beta + \mathbf{Y}_1\mathbf{J})\mathbf{E}^{-1}(\mathbf{Y}_\beta + \mathbf{Y}_1\mathbf{J})^{\mathrm{T}}, \\ \omega\mathbf{E}^{-1} > (\mathbf{Y}_\beta + \mathbf{L}\mathbf{Y}_2)\mathbf{E}^{-1}(\mathbf{Y}_\beta + \mathbf{L}\mathbf{Y}_2)^{\mathrm{T}}. \end{array} \right. \quad (56)$$

The inequality (56) is also an LMI when $\mathbf{L}$ is given. Based on the equivalent relations among (50), (51) and (56), the optimisation problem (49) is solved in this paper by an algorithm similar to the dual iteration algorithm [29]. We tackle the problem (49) in two stages. The first stage's task is to obtain an $\mathbf{L}_{in} \in \mathcal{R}^{(n+m+N+n_1+n_2)\times(t+m)}$ which satisfies

$$\left\{ \begin{array}{l} \mathbf{E}^{-1} > (\mathbf{Y}_0 + \mathbf{Y}_1\mathbf{J})\mathbf{E}^{-1}(\mathbf{Y}_0 + \mathbf{Y}_1\mathbf{J})^{\mathrm{T}}, \\ \mathbf{E}^{-1} > (\mathbf{Y}_0 + \mathbf{L}_{in}\mathbf{Y}_2)\mathbf{E}^{-1}(\mathbf{Y}_0 + \mathbf{L}_{in}\mathbf{Y}_2)^{\mathrm{T}}, \end{array} \right. \quad (57)$$

for some $0 < \mathbf{E} \in \mathcal{E}_{\mathcal{R}\mathcal{K}_\theta}$ and $\mathbf{J} \in \mathcal{R}^{(s+m)\times(n+m+N+n_1+n_2)}$, where $\mathbf{Y}_0$ is the value of $\mathbf{Y}_\beta$ at $\beta = 0$. In the second stage, the problem (49) is solved with the feasible starting point $\mathbf{L}_{in}$. The details are as follows.

Stage 1

Step 1) Set the iterative index $i = 0$ and arbitrarily select an $\mathbf{L}_{(i)} \in \mathcal{R}^{(n+m+N+n_1+n_2)\times(t+m)}$.

Step 2) Solve

$$\inf_{\omega\in\mathcal{R}} \omega, \tag{58}$$
$$\text{s.t.}\quad \omega\mathbf{E}^{-1} > (\mathbf{Y}_0+\mathbf{Y}_1\mathbf{J})\mathbf{E}^{-1}(\mathbf{Y}_0+\mathbf{Y}_1\mathbf{J})^{\mathrm{T}},$$
$$\omega\mathbf{E}^{-1} > (\mathbf{Y}_0+\mathbf{L}_{(i)}\mathbf{Y}_2)\mathbf{E}^{-1}(\mathbf{Y}_0+\mathbf{L}_{(i)}\mathbf{Y}_2)^{\mathrm{T}},$$
$$0<\mathbf{E}\in\mathcal{E}_{\mathcal{RK}_\theta},\ \mathbf{J}\in\mathcal{R}^{(s+m)\times(n+m+N+n_1+n_2)},$$

by a combination of LMI technique and bisection search. Let a minimiser be $\mathbf{J}_{(i)}$.

Step 3) Solve

$$\omega_{i+1} = \inf_{\omega\in\mathcal{R}} \omega, \tag{59}$$
$$\text{s.t.}\ \omega\mathbf{E} > (\mathbf{Y}_0+\mathbf{Y}_1\mathbf{J}_{(i)})^{\mathrm{T}}\mathbf{E}(\mathbf{Y}_0+\mathbf{Y}_1\mathbf{J}_{(i)}),$$
$$\omega\mathbf{E} > (\mathbf{Y}_0+\mathbf{L}\mathbf{Y}_2)^{\mathrm{T}}\mathbf{E}(\mathbf{Y}_0+\mathbf{L}\mathbf{Y}_2),$$
$$0<\mathbf{E}\in\mathcal{E}_{\mathcal{RK}_\theta},\ \mathbf{L}\in\mathcal{R}^{(n+m+N+n_1+n_2)\times(t+m)},$$

by a combination of LMI technique and bisection search. Let a minimiser be $\mathbf{L}_{(i+1)}$.

Step 4) Set $i=i+1$. If $\omega_i>1$, go to *Step 2)*; if $\omega_i\le 1$, let $\mathbf{L}_{in}=\mathbf{L}_{(i)}$ and enter **Stage 2**.

Stage 2

Step 5) Let the iterative index be $i=0$ and $\mathbf{L}_{(i)}=\mathbf{L}_{in}$, and set N_{it} to a sufficiently large integer.

Step 6) Solve

$$\sup_{0\le\beta\in\mathcal{R}} \beta, \tag{60}$$
$$\text{s.t.}\quad \mathbf{E}^{-1} > (\mathbf{Y}_\beta+\mathbf{Y}_1\mathbf{J})\mathbf{E}^{-1}(\mathbf{Y}_\beta+\mathbf{Y}_1\mathbf{J})^{\mathrm{T}},$$
$$\mathbf{E}^{-1} > (\mathbf{Y}_\beta+\mathbf{L}_{(i)}\mathbf{Y}_2)\mathbf{E}^{-1}(\mathbf{Y}_\beta+\mathbf{L}_{(i)}\mathbf{Y}_2)^{\mathrm{T}},$$
$$0<\mathbf{E}\in\mathcal{E}_{\mathcal{RK}_\theta},\ \mathbf{J}\in\mathcal{R}^{(s+m)\times(n+m+N+n_1+n_2)},$$

by the combined LMI technique and bisection search. Let a maximiser be $\mathbf{J}_{(i)}$.

Step 7) Solve

$$\beta_{i+1} = \sup_{0\le\beta\in\mathcal{R}} \beta, \tag{61}$$
$$\text{s.t.}\ \mathbf{E} > (\mathbf{Y}_\beta+\mathbf{Y}_1\mathbf{J}_{(i)})^{\mathrm{T}}\mathbf{E}(\mathbf{Y}_\beta+\mathbf{Y}_1\mathbf{J}_{(i)}),$$
$$\mathbf{E} > (\mathbf{Y}_\beta+\mathbf{L}\mathbf{Y}_2)^{\mathrm{T}}\mathbf{E}(\mathbf{Y}_\beta+\mathbf{L}\mathbf{Y}_2),$$
$$0<\mathbf{E}\in\mathcal{E}_{\mathcal{RK}_\theta},\ \mathbf{L}\in\mathcal{R}^{(n+m+N+n_1+n_2)\times(t+m)},$$

by the combined LMI technique and bisection search. Let a maximiser be $\mathbf{L}_{(i+1)}$, and denote $\mathbf{E}_{(i+1)}$ the corresponding positive definite matrix.

Step 8) Set $i=i+1$. If $i<N_{it}$, go to *Step 6)*; if $i\ge N_{it}$, go to *Step 9)*.

Step 9) Set $\mathbf{E}_d=\mathbf{E}_{(i)}$, denote $\mathbf{Y}_d$ the value of $\mathbf{Y}_\beta$ at $\beta=\beta_i$, and calculate the optimal controller

$$\mathbf{X}_{d\mathrm{opt}} = -(\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}_d\mathbf{Y}_1)^{-1}\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}_d\mathbf{Y}_d\boldsymbol{\Xi}_d\mathbf{Y}_2^{\mathrm{T}} \times(\mathbf{Y}_2\boldsymbol{\Xi}_d\mathbf{Y}_2^{\mathrm{T}})^{-1} \tag{62}$$

with

$$\boldsymbol{\Xi}_d \triangleq \left(\mathbf{E}_d-\mathbf{Y}_d^{\mathrm{T}}\mathbf{E}_d\mathbf{Y}_d+\mathbf{Y}_d^{\mathrm{T}}\mathbf{E}_d\mathbf{Y}_1 \times(\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}_d\mathbf{Y}_1)^{-1}\mathbf{Y}_1^{\mathrm{T}}\mathbf{E}_d\mathbf{Y}_d\right)^{-1}. \tag{63}$$

Step 10) Compute $\widetilde{\upsilon}_d(\mathbf{X}_{d\mathrm{opt}})$ by solving (42) and terminate the routine.

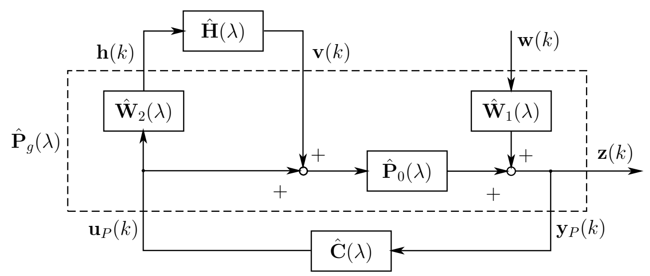

Fig. 1. System configuration of the robust finite-word-length control system design example.

V. A Numerical Design Example

The system configuration for this robust FWL control system design example is shown in Figure 1, where

$$\hat{\mathbf{P}}_0(\lambda)=\frac{\hat{\mathbf{P}}_{n0}(\lambda)}{\hat{\mathbf{P}}_{d0}(\lambda)}$$

with $\hat{\mathbf{P}}_{n0}(\lambda) = 3.3750\times10^{-3}\lambda + 1.3669\times10^{-2}\lambda^2 + 3.4605\times10^{-3}\lambda^3$ and $\hat{\mathbf{P}}_{d0}(\lambda)=1-3.0488\lambda+3.1001\lambda^2 -1.0513\lambda^3$,

$$\hat{\mathbf{W}}_1(\lambda) = \frac{4.9875\times10^{-3}\lambda}{1-9.9501\times10^{-1}\lambda},$$
$$\hat{\mathbf{W}}_2(\lambda) = \frac{5.8512\times10^{-1}\lambda-5.5933\times10^{-1}\lambda^2}{1-1.3390\lambda+3.7908\times10^{-1}\lambda^2},$$

and the plant model uncertainty $\hat{\mathbf{H}}(\lambda)\in\mathcal{H}_\tau$ with $\tau=0.4$. From the given $\hat{\mathbf{P}}_0(\lambda)$, $\hat{\mathbf{W}}_1(\lambda)$ and $\hat{\mathbf{W}}_2(\lambda)$, it was easy to obtain the nominal plant model $\hat{\mathbf{P}}_g(\lambda)$ described by

$$\mathbf{A}_P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1.0513 & -3.1001 & 3.0488 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.99501 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -0.37908 & 1.3390 \end{bmatrix},$$

$$\mathbf{B}_v = \begin{bmatrix} 3.3750\times10^{-3} \\ 2.3959\times10^{-2} \\ 6.6043\times10^{-2} \\ 0 \\ 0 \\ 0 \end{bmatrix},\quad \mathbf{B}_w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4.9875\times10^{-3} \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_P = \begin{bmatrix} 3.3750\times10^{-3} \\ 2.3959\times10^{-2} \\ 6.6043\times10^{-2} \\ 0 \\ 0.58512 \\ 0.22413 \end{bmatrix},$$

$$\mathbf{C}_h = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{C}_z = \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \end{array} \right],$$
$$\mathbf{C}_P = \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \end{array} \right],$$

and

$$\mathbf{D}_{1,1} = 0, \quad \mathbf{D}_{1,2} = 0, \quad \mathbf{D}_{2,1} = 0, \quad \mathbf{D}_{2,2} = 0.$$

For this example, the constant ξ that bounds the closed-loop gain from $\mathbf{w}(k)$ to $\mathbf{z}(k)$ was set to $\xi = 0.3$, and the controller order was chosen to be $m = 2$. The task was thus to design a 2nd-order controller realization directly based on the robust FWL performance measure $\widetilde{\upsilon}_d$.

For this design example, the optimisation problem (49) was formulated and the algorithm described in Section IV was used to find solutions of the optimal robust FWL design problem (49) with an initial guess of $\mathbf{L}_{(0)} = \left[\begin{array}{cc} \mathbf{I}_3 & \mathbf{0} \end{array} \right]^{\mathrm{T}} \in \mathcal{R}^{19\times 3}$ in Stage 1. The resulting controller realization was

$$\mathbf{X}_{d\text{opt}} = \left[\begin{array}{c|cc} -1.0344\times 10^{+2} & -15.600 & -1.4984 \\ \hline -16.070 & -1.4261 & 0.25055 \\ -19.469 & -3.0400 & 0.37517 \end{array} \right]$$

which achieves a robust FWL performance $\widetilde{\upsilon}_d(\mathbf{X}_{d\text{opt}}) = 8.2842\times 10^{-3}$. This designed controller realization achieves the required robust control performance as well as is robust to FWL perturbation errors because, for any FWL perturbation to $\mathbf{X}_{d\text{opt}}$ smaller than 8.2842×10^{-3} and for any $\hat{\mathbf{H}}(\lambda) \in \mathcal{H}_\tau$ with $\tau = 0.4$, the closed-loop system maintains stability and the closed-loop gain from $\mathbf{w}(k)$ to $\mathbf{z}(k)$ is always less than 0.3.

VI. Conclusions

A direct FWL controller design approach has been proposed based on the mixed μ theory, where the task is to design directly an optimal robust FWL controller. A novel robust FWL control performance measure has been proposed which takes into account the standard robust control requirements as well as the FWL implementation considerations. This robust FWL control performance measure can be computed conveniently using an LMI method. The corresponding optimal robust FWL controller design problem has been formulated naturally as a mixed μ problem which can be solved by means of BMI techniques.

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