where ϕ_i is defined as shown in Figure 8.2.1. Equations 8.2.2 can also be obtained by taking the components of

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \tag{8.2.3}$$

where $\boldsymbol{\omega} = \mathbf{k}\omega$.

Let us calculate the kinetic energy of rotation of the body. We have

$$T_{rot} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I_z \omega^2 \tag{8.2.4}$$

where

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2) \tag{8.2.5}$$

The quantity I_z, defined by Equation 8.2.5, is called the *moment of inertia* about the z-axis.

To show how the moment of inertia further enters the picture, let us next calculate the angular momentum about the axis of rotation. Because the angular momentum of a single particle is, by definition, $\mathbf{r}_i \times m_i \mathbf{v}_i$, the z-component is

$$m_i(x_i \dot{y}_i - y_i \dot{x}_i) = m_i(x_i^2 + y_i^2)\omega = m_i r_i^2 \omega \tag{8.2.6}$$

where we have made use of Equations 8.2.2. The total z-component of the angular momentum, which we shall call L_z, is then given by summing over all the particles, namely,

$$L_z = \sum_i m_i r_i^2 \omega = I_z \omega \tag{8.2.7}$$

In Section 7.2 we found that the rate of change of angular momentum for any system is equal to the total moment of the external forces. For a body constrained to rotate about a fixed axis, taken here as the z-axis, then

$$N_z = \frac{dL_z}{dt} = \frac{d(I_z \omega)}{dt} \tag{8.2.8}$$

where N_z is the total moment of all the applied forces about the axis of rotation (the component of $\mathbf{N}$ along the z-axis). If the body is rigid, then I_z is constant, and we can write

$$N_z = I_z \frac{d\omega}{dt} \tag{8.2.9}$$

The analogy between the equations for translation and for rotation about a fixed axis is shown in the following table:

Translation along x-axis		*Rotation about z-axis*	
Linear momentum	$p_x = mv_x$	Angular momentum	$L_z = I_z \omega$
Force	$F_x = m\dot{v}_x$	Torque	$N_z = I_z \dot{\omega}$
Kinetic energy	$T = \frac{1}{2} mv^2$	Kinetic energy	$T_{rot} = \frac{1}{2} I_z \omega^2$

Thus, the moment of inertia is analogous to mass; it is a measure of the rotational inertia of a body relative to some fixed axis of rotation, just as mass is a measure of translational inertia of a body.

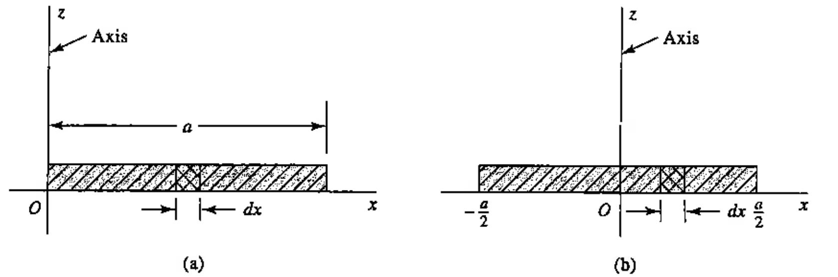

Figure 8.3.1 Coordinates for calculating the moment of inertia of a rod (a) about one end and (b) about the center of the rod.

8.3 | CALCULATION OF THE MOMENT OF INERTIA

In calculations of the moment of inertia $\Sigma\, m_i r_i^2$ for extended bodies, we can replace the summation by an integration over the body, just as we did in calculation of the center of mass. Thus, we may write for any axis

$$I = \int r^2\, dm \tag{8.3.1}$$

where the element of mass dm is given by a density factor multiplied by an appropriate differential (volume, area, or length), and r is the perpendicular distance from the element of mass to the axis of rotation.[1]

In the case of a composite body, it is clear, from the definition of the moment of inertia, that we may write

$$I = I_1 + I_2 + \ldots \tag{8.3.2}$$

where I_1, I_2, and so on, are the moments of inertia of the various parts about the particular axis chosen.

Let us calculate the moments of inertia for some important special cases.

Thin Rod

For a thin, uniform rod of length a and mass m, we have, for an axis perpendicular to the rod at one end (Figure 8.3.1a),

$$I_z = \int_0^a x^2 \rho\, dx = \frac{1}{3}\rho a^3 = \frac{1}{3} m a^2 \tag{8.3.3}$$

The last step follows from the fact that $m = \rho a$.

[1] In Chapter 9, when we discuss the rotational motion of three-dimensional bodies, the distance between the mass element dm and the axis of rotation will be designed $r_\perp$ to remind us that the relevant distance is the one perpendicular to the axis of rotation.

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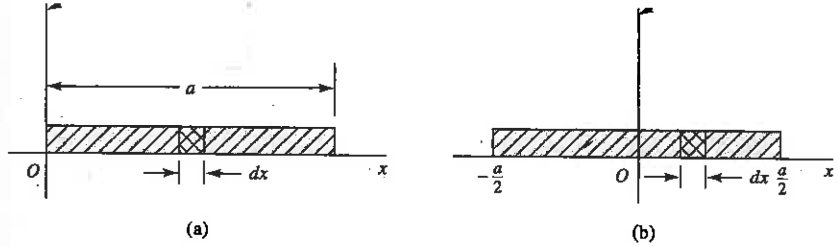

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[1] In Chapter 9, when we discuss the rotational motion of three-dimensional bodies, the distance between the mass element dm and the axis of rotation will be designed $r_\perp$ to remind us that the relevant distance is the one perpendicular to the axis of rotation.

If the axis is taken at the center of the rod (Figure 8.3.1b), we have

$$I_z = \int_{-a/2}^{a/2} x^2 \rho \, dx = \frac{1}{12}\rho a^3 = \frac{1}{12} m a^2 \qquad (8.3.4)$$

Hoop or Cylindrical Shell

In the case of a thin circular hoop or cylindrical shell, for the central, or *symmetry*, axis, all particles lie at the same distance from the axis. Thus,

$$I_{axis} = ma^2 \qquad (8.3.5)$$

where a is the radius and m is the mass.

Circular Disc or Cylinder

To calculate the moment of inertia of a uniform circular disc of radius a and mass m, we shall use polar coordinates. The element of mass, a thin ring of radius r and thickness dr, is given by

$$dm = \rho 2\pi r \, dr \qquad (8.3.6)$$

where ρ is the mass per unit area. The moment of inertia about an axis through the center of the disc normal to the plane faces (Figure 8.3.2) is obtained as follows:

$$I_{axis} = \int_0^a r^2 \rho \, 2\pi r \, dr = 2\pi\rho \frac{a^4}{4} = \frac{1}{2} m a^2 \qquad (8.3.7)$$

The last step results from the relation $m = \rho\pi a^2$.

Clearly, Equation 8.3.7 also applies to a uniform right-circular cylinder of radius a and mass m, the axis being the central axis of the cylinder.

Sphere

Let us find the moment of inertia of a uniform solid sphere of radius a and mass m about an axis (the z-axis) passing through the center. We shall divide the sphere into thin circular discs, as shown in Figure 8.3.3. The moment of inertia of a representative disc of radius y, from Equation 8.3.7, is $\frac{1}{2}y^2\,dm$. But $dm = \rho\pi y^2\,dz$; hence,

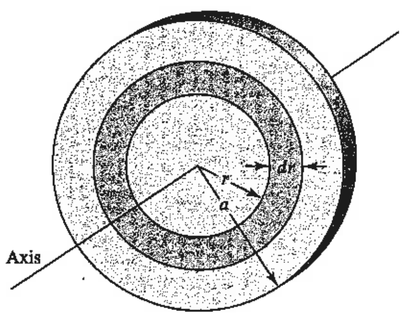

Figure 8.3.2 Coordinates for finding the moment of inertia of a disc.

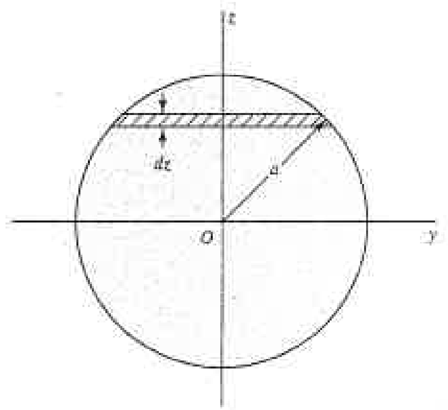

Figure 8.3.3 Coordinates for finding the moment of inertia of a sphere.

$$I_z = \int_{-a}^{a} \frac{1}{2}\pi\rho y^4 \, dz = \int_{-a}^{a} \frac{1}{2}\pi\rho (a^2 - z^2)^2 \, dz = \frac{8}{15}\pi\rho a^5 \qquad (8.3.8)$$

The last step in Equation 8.3.8 should be filled in by the student. Because the mass m is given by

$$m = \frac{4}{3}\pi a^3 \rho \qquad (8.3.9)$$

we have

$$I_z = \frac{2}{5} m a^2 \qquad (8.3.10)$$

for a solid uniform sphere. Clearly also, $I_x = I_y = I_z$.

Spherical Shell

The moment of inertia of a thin, uniform, spherical shell can be found very simply by application of Equation 8.3.8. If we differentiate with respect to a, namely,

$$dI_z = \frac{8}{3}\pi\rho a^4 \, da \qquad (8.3.11)$$

the result is the moment of inertia of a shell of thickness da and radius a. The mass of the shell is $4\pi a^2 \rho \, da$. Hence, we can write

$$I_z = \frac{2}{3} m a^2 \qquad (8.3.12)$$

for the moment of inertia of a thin shell of radius a and mass m. The student should verify this result by direct integration.

EXAMPLE 8.3.1

Shown in Figure 8.3.4 is a uniform chain of length $l = 2\pi R$ and mass $m = M/2$ that is initially wrapped around a uniform, thin disc of radius R and mass M. One tiny piece of chain initially hangs free, perpendicular to the horizontal axis. When the disc is released, the chain falls and unwraps. The disc begins to rotate faster and faster about its fixed z-axis, without friction. (a) Find the angular speed of the disc at the moment the chain completely unwraps. (b) Solve for the case of a chain wrapped around a wheel whose mass is the same as that of the disc, but concentrated in a thin rim.

Solution:

(a) Figure 8.3.4 shows the disc and chain at the moment the chain unwrapped. The final angular speed of the disc is ω. Energy was conserved as the chain unwrapped. Because the center of mass of the chain originally coincided with that of the disc, it fell a distance $l/2 = \pi R$, and we have

$$mg\frac{l}{2} = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$\frac{l}{2} = \pi R \qquad v = \omega R \qquad I = \frac{1}{2}MR^2$$

Solving for ω^2 gives

$$\omega^2 = \frac{mg(l/2)}{[(1/2)(M/2) + (1/2)(m)]R^2} = \frac{mg\pi R}{[(1/2)m + (1/2)m]R^2}$$
$$= \pi\frac{g}{R}$$

(b) The moment of inertia of a wheel is $I = MR^2$. Substituting this into the preceding equation yields

$$\omega^2 = \pi\frac{2g}{3R}$$

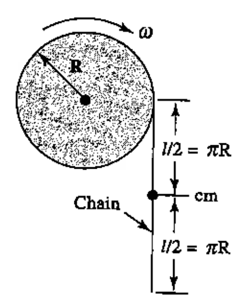

Figure 8.3.4 Falling chain attached to disc, free to rotate about a fixed z-axis.

Even though the mass of the wheel is the same as that of the disc, its moment of inertia is larger, because all its mass is concentrated along the rim. Thus, its angular acceleration and final angular velocity are less than that of the disc.

Perpendicular-Axis Theorem for a Plane Lamina

Consider a rigid body that is in the form of a plane lamina of any shape. Let us place the lamina in the xy plane (Figure 8.3.5). The moment of inertia about the z-axis is given by

$$I_z = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 \qquad (8.3.13)$$

But the sum $\Sigma\, m_i x_i^2$ is just the moment of inertia I_y about the y-axis, because z_i is zero for all particles. Similarly, $\Sigma_i\, m_i y_i^2$ is the moment of inertia I_x about the x-axis. Equation 8.3.13 can therefore be written

$$I_z = I_x + I_y \qquad (8.3.14)$$

This is the perpendicular-axis theorem. In words:

The moment of inertia of any plane lamina about an axis normal to the plane of the lamina is equal to the sum of the moments of inertia about any two mutually perpendicular axes passing through the given axis and lying in the plane of the lamina.

As an example of the use of this theorem, let us consider a thin circular disc in the xy plane (Figure 8.3.6). From Equation 8.3.7 we have

$$I_z = \frac{1}{2}ma^2 = I_x + I_y \qquad (8.3.15)$$

In this case, however, we know from symmetry that $I_x = I_y$. We must therefore have

$$I_x = I_y = \frac{1}{4}ma^2 \qquad (8.3.16)$$

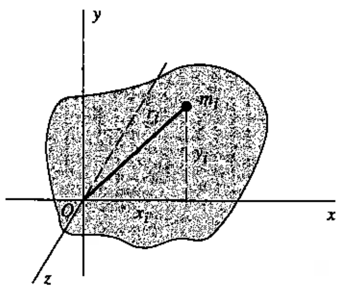

Figure 8.3.5 The perpendicular-axis theorem for a lamina.

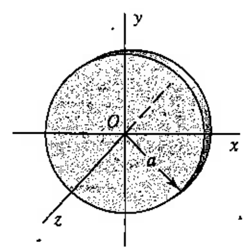

Figure 8.3.6 Circular disc.