[x] [x] [x] previou

Next: solution **Up:** The moment of inertia **Previous:** The moment of inertia

example of a ring

Suppose we have something like a hoola hoop, that is a thin hoop or ring that rotates about the central axis perpendicular to the hoop. What's the moment of inertia in terms of the total mass M of the hoop and its radius R?

Joshua Deutsch
Wed Jan 22 17:07:34 PST 1997

[x] [x] [x] previo

Next: example of a disk **Up:** The moment of inertia **Previous:** example of a ring

solution

This problem is so simple, we can think about it in terms of discrete masses.

$$I = \sum_i m_i r_i^2 \tag{1.34}$$

but all the r_i's are the same and equal to *R*. So we can factor out the radii,

$$I = R^2 \sum_i m_i \tag{1.35}$$

But $\sum m_i$ is just the total mass *M*. Therefore

$$I_{hoop} = MR^2 \tag{1.36}$$

Joshua Deutsch
Wed Jan 22 17:07:34 PST 1997

[x] [x] [x] previou

Next: solution **Up:** The moment of inertia **Previous:** solution

example of a disk

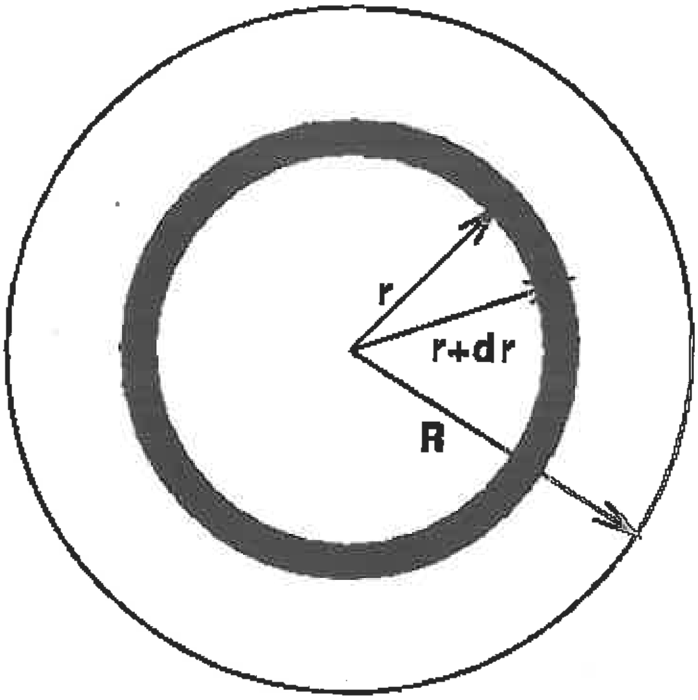
Now we know the moment of inertia of a ring, let's calculate what it is for a disk of uniform density that rotates about the central axis perpendicular to the disk. Call the total mass of the disk M, and its radius R.

Joshua Deutsch
Wed Jan 22 17:07:34 PST 1997

Next: example of a spherical **Up:** The moment of inertia **Previous:** example of a disk

solution

The trick here is to think of the disk as a collection of a bunch of concentric rings. Call the mass per unit area of the disk σ . Then we have a bunch of rings, the inner radius of one of these rings is r and outer radius $r+dr$.



Then the mass of this ring dm is the surface dA times σ . So

$$dm = \sigma dA \quad . \tag{1.37}$$

But dA is just the circumference times dr, $dA = 2\pi r \; dr$. So

$$dI = dm r^2 = \sigma 2\pi r \; dr \; r^2 \tag{1.38}$$

Now we want to integrate dI over all radii to get I so

$$I = \int_0^R dI = \int_0^R \sigma 2\pi r \; r^2 \; dr = \sigma 2\pi \int_0^R r^3 dr = \sigma 2\pi \frac{R^4}{4} \tag{1.39}$$

Let's write σ in terms of M and R:

$$\sigma = \frac{M}{\pi R^2} \tag{1.40}$$

So finally we get

$$I_{disk} = \frac{1}{2} M R^2 \tag{1.41}$$

Joshua Deutsch
Wed Jan 22 17:07:34 PST 1997

[x] [x] [x] previous
Next: solution **Up:** The moment of inertia **Previous:** solution

example of a spherical shell

Calculate the moment of inertia of a spherical shell of mass M and radius R that rotates through an axis that goes throught the center of the sphere.

Joshua Deutsch
Wed Jan 22 17:07:34 PST 1997

[x] [x] [x] previous
Next: example of a solid **Up:** The moment of inertia **Previous:** example of a spherical

solution

So now that you've seen how to make a disk out of a bunch of hoops, we could instead make a spherical shell out of a bunch of then also. It's kind of like a technique in pottery where you slowly add little rings of clay of different sizes, until you have a beautiful vase! Oh shut up!

So we can do that here too. Pottery and physics meet. On the other hand I was never good much at pottery. You get the size of a ring off by a factor of two and it ends up looking like a moldy lump of clay. The same is true of the math involved in this example. I could go through and do it, but it's a bit tedious. There's a much much more elegant way of calculating the moment of inertia in this example. It requires you to think a lot more, but it requires you to write a lot less.

It uses the symmetry of sphere. Let's write things out in terms of discrete masses because it's easier to understand

$$I = \sum_i m_i r_i^2 \qquad (1.34)$$

If we rotate about the z axis, then r_i is the distance between the point and the z axis, so $r_i^2 = x_i^2 + y_i^2$. So

$$I = \sum_i m_i x_i^2 + y_i^2 \qquad (1.42)$$

We could instead compute what I'll call I_x

$$I_x = \sum_i m_i x_i^2 \qquad (1.43)$$

or

$$I_y = \sum_i m_i y_i^2 \qquad (1.44)$$

Because of the symmetry of a sphere we can replace x by y and nothing should change so

$$I_x = I_y \qquad (1.45)$$

I could also calcluate

$$I_z = \sum_i m_i z_i^2 \qquad (1.46)$$

That should also be the same as I_x, again because of symmetry. There is nothing special about the choice of axis. We could call x y, y z, and z x, and we'd get the same answers.

Now lets calculate $3I_x = I_x + I_y + I_z$. That's

$$\sum_i m_i x_i^2 + \sum_i m_i y_i^2 + \sum_i m_i z_i^2 = \sum_i m_i (x_i^2 + y_i^2 + z_i^2) \qquad (1.47)$$

But since we have a sphere, we know that $x_i^2 + y_i^2 + z_i^2 = R^2$. So we can pull that out of the sum and then we just have a sum over the m_i's which just equals *M*. So $3I_x = MR^2$. But $I = I_x + I_y = 2I_x$. So

$$I_{shell} = \frac{2}{3}MR^2 \qquad (1.48)$$

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[x] [x] [x] previou
Next: example of a solid **Up:** The moment of inertia **Previous:** example of a spherical

Joshua Deutsch
Wed Jan 22 17:07:34 PST 1997

[x] r [x] [x] previou
Next: solution **Up:** The moment of inertia **Previous:** solution

example of a solid sphere

What's the moment of inertia of a solid sphere through an axis that passes through its center? The sphere is of uniform density.

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[x] [x] [x] previo

Next: Cool Theorems about I **Up:** The moment of inertia **Previous:** example of a solid

solution

Now we have the moment of inertia of a spherical shell, we can sum up all these shells to get what it is for solid sphere. This is a lot like the example of the disk.

So what's the mass *dm* of a shell of inner radius *r* and outer radius *r+dr*? Call the density ρ. Then $dm = \rho dV$. What's the volume *dV*? It's the surface area of a sphere of radius *r* times *dr*. The surface area of a sphere is $4\pi r^2$ so

$$dm = \rho 4\pi r^2 dr \tag{1.49}$$

And from the last example, that $dI = (2/3)\ dm\ r^2$. So the moment of inertia is

$$I = \int dI = \int \frac{2}{3} dm r^2 = \int_0^R \frac{2}{3}(\rho 4\pi r^2) dr r^2 = \frac{2}{3}\rho 4\pi \int_0^R r^4 dr = \frac{2}{3} 4\pi\rho \frac{R^5}{5} \tag{1.50}$$

Let's write ρ in terms of the *M* and *R*. The volume a sphere is $\frac{4}{3}\pi R^3$, so

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \tag{1.51}$$

Plugging that in to the formula for *I*

$$I = \frac{2}{3} 4\pi \frac{M}{\frac{4}{3}\pi R^3} \frac{R^5}{5} \tag{1.52}$$

or

$$I_{sphere} = \frac{2}{5} M R^2 \tag{1.53}$$

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Wed Jan 22 17:07:34 PST 1997