Derivation of the radial and angular equations of motion: page 27 of the lecture notes

First, as we have already shown that planetary motion is planar (e.g., on the xy Cartesian plane), write the gravitational force $\mathbf{F}_g$ between the sun (with mass M) and the planet (with mass m) and the position vector $\mathbf{r}$ of the latter with the respect to the former in polar coordinates,

$$\mathbf{F}_g = F_g \cos\theta \hat{\mathbf{x}} + F_g \sin\theta \hat{\mathbf{y}} \qquad \mathbf{r} = r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}}. \tag{1}$$

(Notice that r and θ are functions of times, while $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are not, as they are unit vectors always pointing along the positive x and y axis, respectively). In doing so, recall that we have placed the origin of coordinates in the CM of the sun-planet system, which coincides with the geometrical centre of the sun, assumed to be a uniform spherical object, in our usual approximation $M \gg m$.
By recalling the equation of motion in cartesian coordinates

$$\mathbf{F}_g = m\frac{d^2\mathbf{r}}{dt^2} \tag{2}$$

and using the second of the relations in (1), we obtain

$$\begin{aligned}
m\frac{d^2\mathbf{r}}{dt^2} &= m\frac{d^2}{dt^2}(r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}}) \qquad (3)\\
&= m\frac{d}{dt}(\dot{r}\cos\theta\hat{\mathbf{x}} - r\sin\theta\dot{\theta}\hat{\mathbf{x}} + \dot{r}\sin\theta\hat{\mathbf{y}} + r\cos\theta\dot{\theta}\hat{\mathbf{y}})\\
&= m(\ddot{r}\cos\theta\hat{\mathbf{x}} - \dot{r}\sin\theta\dot{\theta}\hat{\mathbf{x}} - \dot{r}\sin\theta\dot{\theta}\hat{\mathbf{x}} - r\cos\theta\dot{\theta}^2\hat{\mathbf{x}} - r\sin\theta\ddot{\theta}\hat{\mathbf{x}}\\
&\quad +\ddot{r}\sin\theta\hat{\mathbf{y}} + \dot{r}\cos\theta\dot{\theta}\hat{\mathbf{y}} + \dot{r}\cos\theta\dot{\theta}\hat{\mathbf{y}} - r\sin\theta\dot{\theta}^2\hat{\mathbf{y}} + r\cos\theta\ddot{\theta}\hat{\mathbf{y}})\\
&= m(\ddot{r}\cos\theta\hat{\mathbf{x}} - 2\dot{r}\sin\theta\dot{\theta}\hat{\mathbf{x}} - r\cos\theta\dot{\theta}^2\hat{\mathbf{x}} - r\sin\theta\ddot{\theta}\hat{\mathbf{x}}\\
&\quad +\ddot{r}\sin\theta\hat{\mathbf{y}} + 2\dot{r}\cos\theta\dot{\theta}\hat{\mathbf{y}} - r\sin\theta\dot{\theta}^2\hat{\mathbf{y}} + r\cos\theta\ddot{\theta}\hat{\mathbf{y}}).
\end{aligned}$$

Now, recalling eq. (2), equate the terms proportional to $\cos\theta\hat{\mathbf{x}}$ in the first term of eq. (1) and in eq. (3), to obtain

$$m(\ddot{r} - r\dot{\theta}^2)\cos\theta\hat{\mathbf{x}} = F_g\cos\theta\hat{\mathbf{x}}. \tag{4}$$

Or, equivalently, equate the terms proportional to $\sin\theta\hat{\mathbf{y}}$, to obtain

$$m(\ddot{r} - r\dot{\theta}^2)\sin\theta\hat{\mathbf{y}} = F_g\sin\theta\hat{\mathbf{y}}. \tag{5}$$

As these two last relations ought to be true for each θ, one obtains

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{mr^2} \qquad \text{(radial equation)}, \tag{6}$$

as $F_g = -k/r^2$, with $k = GMm$. As this last equation contains the source of the motion, i.e., the gravitation force, it can effectively be regarded as the equation of motion in planar polar coordinates.
Next, also equate the terms proportional to $\sin\theta\hat{\mathbf{x}}$, to obtain

$$(-2\dot{r}\dot{\theta} - r\ddot{\theta})\sin\theta\hat{\mathbf{x}} = 0. \tag{7}$$

Or, equivalently, equate the terms proportional to $\cos\theta\hat{\mathbf{y}}$, to obtain

$$(2\dot{r}\dot{\theta} + r\ddot{\theta})\cos\theta\hat{\mathbf{y}} = 0. \tag{8}$$

Again, as these two last relations ought to be true for each θ, one obtains

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0. \tag{9}$$

The meaning of this last equation is not immediately evident. However, we can play a 'mathematical trick' to render it a bit more explanatory. Let us multiply and divide it by r both sides of it (we can do this, as r is a finite non-zero quantity) to obtain

$$\frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \qquad \text{(angular equation)}. \tag{10}$$

Thus, as this last equation ought to be true for every r, it expresses the conservation (over time) of some quantity proportional to $r^2\dot{\theta}$. But, what is this conserved quantity ? Recall that, in our system, the total angular momentum $\mathbf{L}$ is conserved. Furthermore, we have already shown that, for our case, the latter is perpendicular to the plane of motion, pointing in the same direction as the vector angular velocity $\boldsymbol{\omega}$. Hence, we can exploit the scalar relation

$$L = I\omega, \tag{11}$$

where I is the moment of inertia of the planet with respect to the axis going through the sun along the direction individuated by $\boldsymbol{\omega}$ (or, equivalently in our case, by $\mathbf{L}$). Explicitly then,

$$L = I\omega = mr^2\dot{\theta}, \tag{12}$$

where we have introduced $I = mr^2$ (as it is the case here) and used the definition $\omega \equiv \dot{\theta}$. As a consequence, if one multiplies and divides eq. (10) by m (entirely legal procedure, as m is some finite non-zero number) and exploits that fact the latter is constant over time, one obtains

$$\frac{1}{mr}\frac{d}{dt}(mr^2\dot{\theta}) = \frac{1}{mr}\frac{dL}{dt} = 0, \tag{13}$$

which implies the conservation of the total angular momentum, i.e.,

$$\frac{dL}{dt} = 0. \tag{14}$$

Notice that the equation of motion in (planar) Cartesian coordinates in (2) has two degrees of freedom, i.e., it is true along both the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions. Therefore, you should not be surprised to have ended up with exactly two independent equations in (planar) polar coordinates, eqs. (6) and (10).

Finally, it is essential for the above derivations to recall the following derivative rule:

$$\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)}\frac{dg(x)}{dx}. \tag{15}$$

Now, why do not try yourself to obtain the expression of the total energy of the system using planar polar coordinates ? That is, starting from the first of the two equations (3.2) in the lecture notes (in which you can set $\mu = m$), to obtain the following equation,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - k/r \tag{16}$$

appearing in page 27 of the lecture notes. Recall that the gravitational potential energy is $V(r) = -k/r$.