Derivation of the solution to the radial equation of motion: page 31 of the lecture notes

Start from

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{mr^2} \qquad \text{(radial equation)},$$

with $k = GMm$. Then eliminate $\dot{\theta}$ using angular momentum conservation, $\dot{\theta} = L/mr^2$, leading to a differential equation for r alone:

$$\ddot{r} - \frac{L^2}{m^2r^3} = -\frac{k}{mr^2}.$$

Use the relation

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} = \frac{L}{mr^2}\frac{d}{d\theta},$$

to obtain derivatives with respect to θ in place of time derivatives, then set

$$r = \frac{1}{u}.$$

This leads to the following results

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{du}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -\frac{L}{m}\frac{du}{d\theta}.$$

The last step follows from, $\dot{\theta} = u^2L/m = L/mr^2$. Differentiating a second time we have

$$\ddot{r} = -\frac{L}{m}\frac{d}{dt}\frac{du}{d\theta} = -\frac{L}{m}\dot{\theta}\frac{d^2u}{d\theta^2} = -u^2\frac{L^2}{m^2}\frac{d^2u}{d\theta^2}.$$

From these values of r, $\ddot{r}$ and $\dot{\theta}$, we find that the radial equation of motion above transforms to

$$\frac{d^2u}{d\theta^2} + u = \frac{mk}{L^2},$$

upon multiplying both sides by $m^2r^2/L^2 = m^2/u^2L^2$ and changing sign.
The solution of the orbit equation is

$$u = \frac{1}{r} = \frac{mk}{L^2}(1 + e\cos\theta),$$

which for any constant e. As done in the lectures, if you replace the solution in the equation immediately above you will see explicitely that the latter is satisfied for any e. For $0 \leq e < 1$ the solution describes an ellipse with semi latus rectum $l = L^2/mk$.