PHYSC000

If not marked, consider original

$\boxed{B = \text{bookwork}}$ — i.e. lect notes, discussed in lects, Prob Sheet Qs etc.

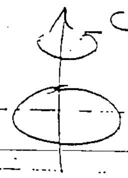
$\boxed{(B) = \text{partly bookwork}}$

(A1) $\underline{R} = \frac{1}{M} \sum_i m_i \underline{r}_i$ where $M = \sum_i m_i$ [1]B

The 1st term contributing to T can be written $\frac{1}{2} M \dot{R}^2$. This is the kinetic energy of the motion of the centre of mass, [1]B
where M is the total mass of the object [1]B
The second term is the kinetic energy of the constituents in the centre of mass frame. It is thus the internal kinetic energy of the system and is the same in all frames. [1]B

(for relative correct)

(A2)



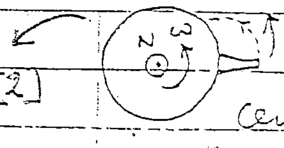
$\omega = \dot{\varphi}$ [2](B)

$K.E = \frac{1}{2} I \omega^2$ [2](B)

(A3) The apparent gravity $g^*$ takes into account the centrifugal force term of the rotating frame: [2]B
$$\underline{g}^* = \underline{g} - \underline{\omega} \times (\underline{\omega} \times \underline{R}),$$
where $\underline{R}$ is position of object measured from the centre the Earth and $\omega$ is the angular velocity of the Earth. It defines a local apparent direction [2]B

(For full credit correct)

(A4) Looking down from the N pole, the ball is projected tangentially when viewed from an inertial frame. Gravitational force is central so angular momentum is conserved. [2]B



[2]

Therefore as ball falls, its angular velocity increases

(A5) If $x(t)$ is a time-dependent solution of the harmonic motion, then Time Translation Invariance states that also $x(t+c)$ is a solution, for any constant $c$

[2] B

B1

a)

t $\quad\quad\quad\quad$ t+δt


2 B

Conservation of momentum of isolated system:

$$m^* v^* = (m^* + \delta m^*)(v^* + \delta v^*) - \delta m^* (v^* - u)$$ 2 B

To first order: $\quad 0 = m^* \delta v^* + \delta m^* u$

Turn into differential:

$$dv^* = -u \frac{dm^*}{m^*} \quad (A)$$ 2 B

Integrate

$$v = \int_0^v dv^* = -u \int_M^m \frac{dm^*}{m^*} = u \ln\left(\frac{M}{m}\right)$$ 2 B [8]

b) Rewrite (A) with explicit t dependence: 2

$$m^* \frac{dv^*}{dt^*} = -u \frac{dm^*}{dt^*} - m^* g$$ 2

$$\Rightarrow \quad v' = \int_0^{v'} dv^* = u \ln\left(\frac{M}{m'}\right) - gt'$$ 2

[6]

c) $\quad v = v' \Rightarrow \quad u \ln\left(\frac{M}{m}\right) = u \ln\left(\frac{M}{m'}\right) - gt'$ 2

$$\Rightarrow \quad \cancel{u \ln(M)} - u \ln(m) - \cancel{u \ln(M)} + u \ln(m') = -gt'$$

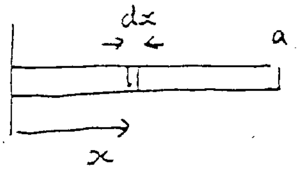
$$\Rightarrow \quad +u \ln\left(\frac{m}{m'}\right) = +gt'$$ 2

$g, u, t'$ positive definite $\Rightarrow \quad m > m'$ 2

[6]

B2

(a)




mass $\rho$ per unit length

$$I = \int_0^a x^2 . \rho\, dx = \frac{\rho a^3}{3} = \boxed{\frac{1}{3} m a^2} \quad [6]$$

(2) (2) (2)

(b) $I_{\text{wheel about centre}} = M a^2 + \frac{n}{3} m a^2$ [3]

↑ rim (1) ↑ spokes (2)

by ‖el axes theorem

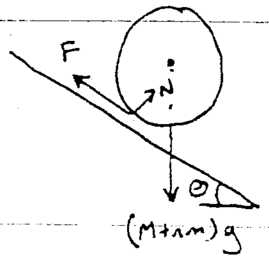
$$I_{\text{about rim}} = \left(M + \frac{nm}{3}\right) a^2 + (M + nm) a^2 = \boxed{\frac{1}{3}(6M + 4nm) a^2} \quad [3]$$

(1) (2)

(c) acceleration down ramp.

use torques about CM $\quad F a = I_{CM} \dot{\omega}$ $\quad [2]_B$

cpt. of linear eq. motion parallel to plane




$$-F + (M+nm) g \sin\theta = (M+nm) a\dot{\omega} \quad [2]_B$$

↑ use no-slip condition.

eliminate F

$$(M+nm) g \sin\theta = a\dot{\omega}\left(M + nm + \frac{I_{CM}}{a^2}\right) \quad (2)$$

$$= a\dot{\omega}\left(M + nm + \frac{\text{[redacted]}M + \text{[redacted]}nm}{3}\right)$$

$$\Rightarrow \text{linear accel} = a\dot{\omega} = \frac{3(M+nm) g \sin\theta}{(6M + 4nm)} \quad (2) \quad [4]_B$$

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Or use energy conservation:

$$\frac{1}{2}(M+nm)\dot{x}^2 + \frac{1}{2}\cdot\frac{1}{3}(3M+nm) a^2 \frac{\dot{x}^2}{a^2} - (M+nm) g \sin\theta\, x = \text{const.}$$

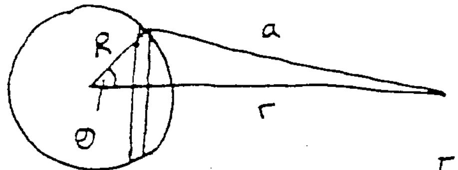
where $x$ is dist rolled down plane.

Differentiate wrt $t$

$$\frac{1}{3}(6M + 4nm)\dot{x}\ddot{x} = (M+nm) g \sin\theta\, \dot{x}$$

same answer

B3 (a) Direct calculation: let $\rho$ = surface mass density




$$d\Phi = -\frac{G\rho\, 2\pi R\sin\theta \,.\, R\,d\theta}{a}$$ [3] B

$$\Phi = -2\pi G\rho R^2 \int_0^{\pi} \frac{\sin\theta\, d\theta}{(r^2+R^2-2Rr\cos\theta)^{1/2}}$$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$
and swap limits

$$= -2\pi G\rho R^2 \int_{-1}^{1} \frac{du}{(r^2+R^2-2Rru)^{1/2}}$$

$$= -2\pi G\rho R^2 \left[\frac{2}{-2Rr}\right]\left[(r^2+R^2-2Rru)^{1/2}\right]_{-1}^{1}$$ [3] B

$$= \frac{2\pi G\rho R}{r}\left(|r-R| - (r+R)\right)$$

So,
$$\boxed{\Phi(r) = \begin{cases} -\dfrac{GM}{r} & r>R \\ -\dfrac{GM}{R} & r<R \end{cases}}$$ using $M = 4\pi\rho R^2$ [2] B

[2] B

(b) From above, see that grav. field outside shell has magnitude $\frac{GM_{shell}}{r^2}$, and inside has magnitude zero. [2] B

Immediately see that magnitude $g(r)$ of grav. field at distance $r$ from centre of a spherically symmetric earth is

$$g(r) = \frac{GM(r)}{r^2}$$ where $M(r)$ is mass within radius $r$. [2]

If $\rho(r)$ is density at radius $r$, then $M(r) = 4\pi\int_0^r \rho(u)\,u^2\,du$

Now: $$\frac{dg}{dr} = \frac{G}{r^2}\frac{dM(r)}{dr} - \frac{2GM(r)}{r^3} = \frac{G}{r^2}\left(4\pi r^2\rho(r) - \frac{2M(r)}{r}\right)$$

we want $\frac{dg}{dr} < 0$ at $r=R$ where $R$ is earth's radius.

so: $$4\pi\rho(R) < \frac{2M(R)}{R^3}$$ but $\rho(R) = \rho_s$, $M(R) = \frac{4}{3}\pi R^3 \rho_{ave}$ [6]

so: $$\boxed{\rho_s < \frac{2}{3}\rho_{ave}}$$

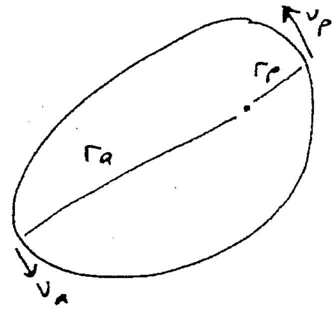
B4) ☐ Gravitational force is central so $\underline{L} = \underline{r} \times \underline{p}$ is a conserved vector.
$\underline{r}$ and $\underline{p}$ must always lie in the plane perp. to $\underline{L}$. [4]

☐ Energy is also conserved (central forces are conservative) [2]

[☐ Also Runge-Lenz vector giving orientation of orbit in its plane — but this is not expected in answer]

☐ Ang. mom: $v_p r_p = v_a r_a$ (1) [2]

Energy: $\dfrac{v_p^2}{2} - \dfrac{GM}{r_p} = \dfrac{v_a^2}{2} - \dfrac{GM}{r_a}$ (2) [2]




start with (2)

$$v_p^2 - v_a^2 = 2GM\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

use (1)

$$v_p^2\left(1 - \frac{r_p^2}{r_a^2}\right) = 2GM\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

$$v_p^2\frac{(r_a^2 - r_p^2)}{r_a^2} = \frac{2GM}{r_p r_a}(r_a - r_p) \Rightarrow v_p^2 = \frac{2GM\, r_a}{(r_a + r_p) r_p} \quad [4]$$

Likewise: $v_a^2 = \dfrac{2GM\, r_p}{(r_a + r_p) r_a}$ [2]

Use $r_a + r_p = 2a$. Hence [2] for this and evidence of realising that no radial component of velocity at perihelion and aphelion.

$$v_p^2 v_a^2 = \frac{2GM}{(r_a + r_p)} \cdot \frac{r_a}{r_p} \cdot \frac{2GM}{(r_a + r_p)} \cdot \frac{r_p}{r_a}$$

$$\boxed{v_p v_a = \frac{GM}{a}} \quad [2]$$