4

Rotating Coordinate Systems

4.1 Time Derivatives in a Rotating Frame

First recall the result that, for a vector $\mathbf{A}$ of *fixed length*, rotating about the origin with constant angular velocity $\boldsymbol{\omega}$, the rate of change of $\mathbf{A}$ is

$$\boxed{\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}}.$$

Now let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ be unit vectors of an inertial frame O and let $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ be unit vectors of a rotating frame O'. Each of the primed basis vectors rotates rigidly with O', so

$$\frac{d\hat{\mathbf{i}}'}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{i}}',$$

with similar equations for $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$. Consider an arbitrary vector $\mathbf{a}$ and resolve it into components in O and O':

$$\mathbf{a} = a_i\hat{\mathbf{i}} + a_j\hat{\mathbf{j}} + a_k\hat{\mathbf{k}} = a_i'\hat{\mathbf{i}}' + a_j'\hat{\mathbf{j}}' + a_k'\hat{\mathbf{k}}'.$$

Differentiating with respect to time gives:

$$\begin{aligned}
\frac{d\mathbf{a}}{dt} &= \frac{da_i}{dt}\hat{\mathbf{i}} + \frac{da_j}{dt}\hat{\mathbf{j}} + \frac{da_k}{dt}\hat{\mathbf{k}} \\
&= \frac{da_i'}{dt}\hat{\mathbf{i}}' + \frac{da_j'}{dt}\hat{\mathbf{j}}' + \frac{da_k'}{dt}\hat{\mathbf{k}}' + a_i'\boldsymbol{\omega} \times \hat{\mathbf{i}}' + a_j'\boldsymbol{\omega} \times \hat{\mathbf{j}}' + a_k'\boldsymbol{\omega} \times \hat{\mathbf{k}}'.
\end{aligned}$$

At this point, we introduce some new notation. We normally use $\dot{\mathbf{a}}$ and $d\mathbf{a}/dt$ interchangeably. Let us now adopt the convention that

$$\dot{\mathbf{a}} \equiv \frac{da_i'}{dt}\hat{\mathbf{i}}' + \frac{da_j'}{dt}\hat{\mathbf{j}}' + \frac{da_k'}{dt}\hat{\mathbf{k}}',$$

which means that you differentiate the *components* of $\mathbf{a}$ but not the basis vectors, even if the basis vectors are time dependent. In other words, $\dot{\mathbf{a}}$ is the *rate of change of* $\mathbf{a}$ *measured in the rotating frame*. The *total* rate of change of $\mathbf{a}$ is then:

$$\boxed{\frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} + \boldsymbol{\omega} \times \mathbf{a}}.$$

There is one term for the rate of change with respect to the rotating axes and a second term arising from the rotation of the axes themselves.

4.2 Equation of Motion in a Rotating Frame

We can use the result we just derived to work out the equation of motion for a particle when its coordinates are measured in a frame rotating at *constant* angular velocity $\boldsymbol{\omega}$. Let $\mathbf{a}$ be a position vector $\mathbf{r}$. Differentiating once:

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}.$$

Differentiating again:

$$\begin{aligned} \frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \ddot{\mathbf{r}} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \end{aligned}$$

Newton's law of motion is $\mathbf{F}_{\text{tot}} = m\,d^2\mathbf{r}/dt^2$, where $\mathbf{F}_{\text{tot}}$ is the total force acting, so the equation of motion in the rotating frame becomes:

$$\boxed{m\ddot{\mathbf{r}} = \mathbf{F}_{\text{tot}} - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}.$$

The last two terms on the right hand side are *apparent* (or *inertial* or *fictitious*) forces, arising because we are measuring positions with respect to axes which are themselves rotating (i.e. accelerating).

4.3 Motion Near the Earth's Surface

Assume that the Earth is spherically symmetric so that the weight of an object is a vector directed towards the Earth's centre. Pick an inertial frame O with origin at the Earth's centre, together with a frame O' also with origin at the Earth's centre, but rotating with the Earth at angular velocity $\boldsymbol{\omega}$. Write the total force on the particle as its weight $m\mathbf{g}$ plus any other external forces $\mathbf{F}$ ($\mathbf{F}_{\text{tot}} = \mathbf{F} + m\mathbf{g}$).

Let $\mathbf{R}$ be a vector from the centre of the Earth to some point on or near its surface, as shown in figure 4.1, and let $\mathbf{x}$ be the displacement of the particle relative to this point. This says that the position vector in O' can be written as

$$\mathbf{r} = \mathbf{R} + \mathbf{x}.$$

Since $\mathbf{R}$ is fixed in O', $\dot{\mathbf{R}} = 0$ and $\ddot{\mathbf{R}} = 0$, and the equation of motion becomes:

$$m\ddot{\mathbf{x}} = \mathbf{F} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \dot{\mathbf{x}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{R} + \mathbf{x}]).$$

We will now drop all terms of order x/R or smaller. Even if x is 10 km, this ratio is $10\,\text{km}/6400\,\text{km} \approx 1.6 \times 10^{-3}$. With this approximation:

1. $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{R} + \mathbf{x}]) \longrightarrow \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$ (If R was not so large we would normally drop this $O(\omega^2)$ term),

2. the term involving $\mathbf{g}$ simplifies,

$$\mathbf{g} = -\frac{GM}{|\mathbf{R} + \mathbf{x}|^3}(\mathbf{R} + \mathbf{x}) \longrightarrow -\frac{GM}{R^3}\mathbf{R} = -g\frac{\mathbf{R}}{R}.$$

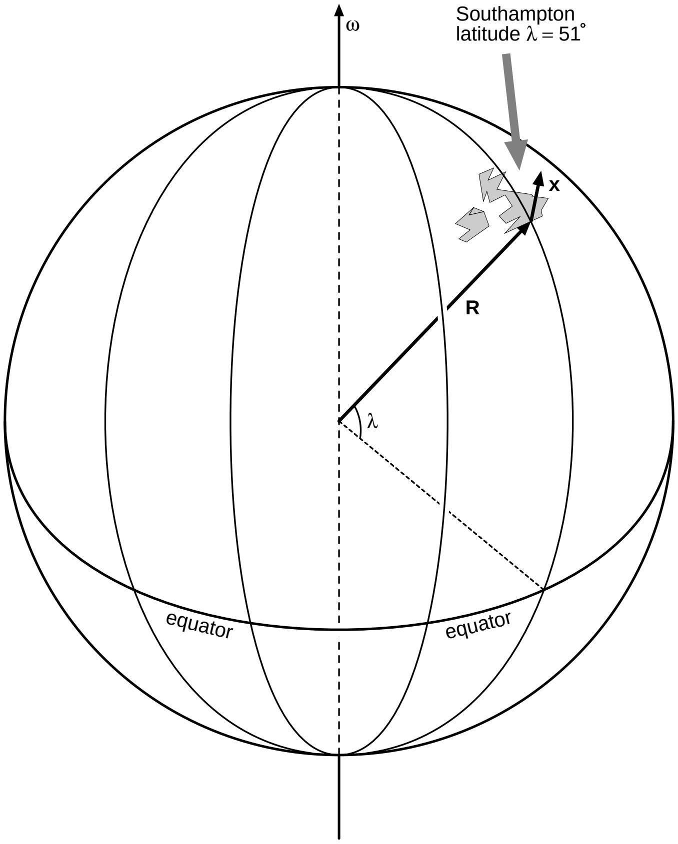

Figure 4.1 Motion near the surface of the Earth. Displacement **x** measured from tip of a (rotating) vector **R** from the Earth's centre to a point on or near its surface.

The approximate equation of motion becomes,

$$\boxed{m\ddot{\mathbf{x}} = \mathbf{F} + m\mathbf{g}^* - 2m\boldsymbol{\omega}\times\dot{\mathbf{x}}},$$

where we have defined the *apparent gravity*,

$$\boxed{\mathbf{g}^* = -g\frac{\mathbf{R}}{R} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})}.$$

We will take the latitude to be λ, as shown in figure 4.1 (note that latitude is zero at the equator).

4.3.1 Apparent Gravity

The apparent gravity $\mathbf{g}^*$ defines a local apparent vertical direction. It is what is measured by hanging a mass from a spring so that the mass is stationary in the rotating frame fixed to the Earth, and $\dot{\mathbf{x}} = 0$, $\ddot{\mathbf{x}} = 0$. We can easily work out the small deflection angle α between the apparent vertical and the true vertical defined by line to the Earth's centre. The situation is illustrated in figure 4.2.

The magnitude of the centrifugal term is,

$$|-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})| = \omega^2 R\cos\lambda.$$

Applying the cosine rule to the right hand triangle in figure 4.2 gives,

$$g^{*2} = g^2 + (\omega^2 R\cos\lambda)^2 - 2g\omega^2 R\cos^2\lambda,$$

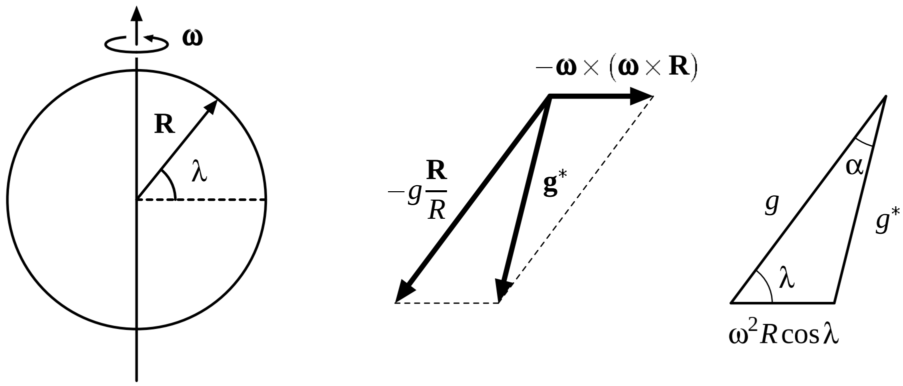

Figure 4.2 Determining the deflection angle between true and apparent verticals on the Earth's surface.

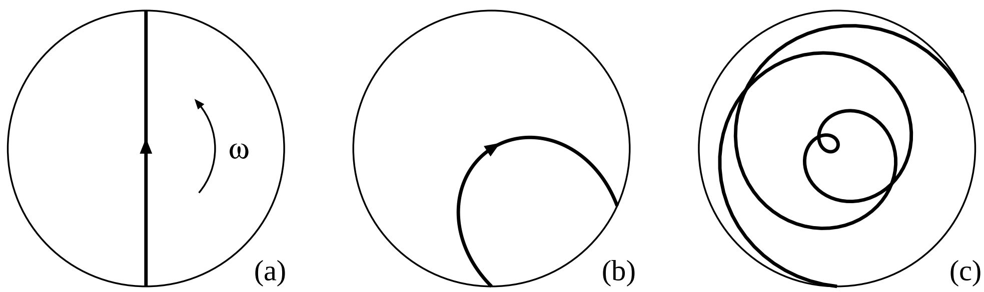

Figure 4.3 Particle moving across a rotating disc: seen from (a) an inertial frame, (b) a frame rotating with the disc, (c) a frame rotating with the disc when $\omega a/v$ is large, where v is the particle's speed in the inertial frame and a is the disc's radius.

which tells us that $g^* = g + O(\omega^2)$. Applying the sine rule to the same triangle gives,

$$\frac{\sin\alpha}{\omega^2 R\cos\lambda} = \frac{\sin\lambda}{g^*}.$$

Since α is small, we approximate $\sin\alpha \approx \alpha$, and to order ω^2 we can replace g^* by g, to find:

$$\boxed{\alpha = \frac{\omega^2 R}{g}\sin\lambda\cos\lambda}.$$

This tells us that the deflection vanishes at the equator and the poles, and is maximal at latitude 45°. The size of the deflection is governed by

$$\frac{\omega^2 R}{g} = \frac{3.4\,\mathrm{cm\,s^{-2}}}{g} = 0.35\%.$$

At Southampton, $\lambda = 51°$, we find $\alpha = 1.7 \times 10^{-3}\,\mathrm{rad} = 0°6'$.

4.3.2 Coriolis Force

The Coriolis "force" (in quotation marks because it's a fictitious or inertial force associated with our use of an accelerated frame) is the term

$$-2m\boldsymbol{\omega} \times \dot{\mathbf{x}}$$

in the equation of motion. You see that it acts at right angles to the direction of motion, and is proportional to the speed. To understand the physical origin of this

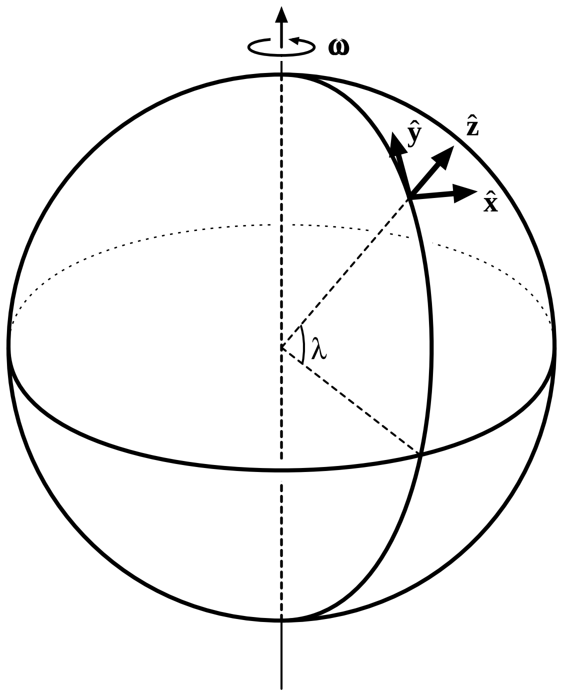

Figure 4.4 Coordinate system on the Earth's surface.

force, it may be helpful to consider a particle moving diametrically across a smooth flat rotating disc with no forces acting horizontally. An observer in an inertial frame (watching the disc from above) will simply see the particle move in a straight line at constant speed, as in figure 4.3(a). However, an observer rotating with the disc will see the particle follow a curved track as in figure 4.3(b). If the observer does not realise that the disc is rotating they will conclude that some force acts on the particle at right angles to its velocity: this is the Coriolis force (in this example, the rotating observer also sees the effect of the apparent force $m\omega^2\mathbf{x}$ acting radially outwards). As the rotation rate, ω, gets large, the path seen by the rotating observer can get quite complicated, figure 4.3(c).

To study the Coriolis force quantitatively, it is helpful to choose a convenient set of axes on the Earth's surface. This is done as follows, and is illustrated in figure 4.4. We choose $\hat{\mathbf{z}}$ along the apparent upward vertical (parallel to $-\mathbf{g}^*$), and take $\hat{\mathbf{x}}$ pointing to the East. The third unit vector $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ therefore points North. Using this coordinate system, the equations of motion are:

$$
\begin{aligned}
m\ddot{x} &= F_x - 2m\omega\left(\dot{z}\cos\lambda - \dot{y}\sin\lambda\right), \\
m\ddot{y} &= F_y - 2m\omega\dot{x}\sin\lambda, \\
m\ddot{z} &= F_z - mg^* + 2m\omega\dot{x}\cos\lambda.
\end{aligned}
\tag{4.1}
$$

4.3.3 Free Fall — Effects of Coriolis Term

For a particle in free fall, the non-gravitational force $\mathbf{F}$ disappears from the equation of motion, which becomes,

$$\ddot{\mathbf{x}} = \mathbf{g}^* - 2\,\boldsymbol{\omega} \times \dot{\mathbf{x}}.$$

We will work to $O(\omega)$ in this section, so we can approximate $\mathbf{g}^*$ by $\mathbf{g}$.

We could investigate this using the coordinate form of the equation of motion given in equation (4.1). However, in this case, we can proceed vectorially and solve all three coordinate equations at the same time.

The equation of motion can be integrated once with respect to time, with the initial conditions $\mathbf{x} = \mathbf{a}$ and $\dot{\mathbf{x}} = \mathbf{v}$ at $t = 0$, corresponding to a particle projected with velocity $\mathbf{v}$ from point $\mathbf{a}$. This gives,

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{g}t - 2\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a}).$$

Since we are ignoring terms of $O(\omega^2)$, we can substitute the zeroth order solution, $\mathbf{x} = \mathbf{a} + \mathbf{v}t + \mathbf{g}t^2/2$ in the cross product term, giving,

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{g}t - 2\boldsymbol{\omega} \times \left(\mathbf{v}t + \frac{1}{2}\mathbf{g}t^2\right).$$

This can be integrated once more, using the same initial conditions, $\mathbf{x} = \mathbf{a}$ and $\dot{\mathbf{x}} = \mathbf{v}$ at $t = 0$, to give:

$$\boxed{\mathbf{x} = \mathbf{a} + \mathbf{v}t + \frac{1}{2}\mathbf{g}t^2 - \boldsymbol{\omega} \times \left(\mathbf{v}t^2 + \frac{1}{3}\mathbf{g}t^3\right)}.$$

Now that we have our solution, we can express it in terms of our choice of coordinates in figure 4.4. We will consider two cases: a particle dropped from a tower and a shell fired from a cannon.

Particle dropped from a tower Consider a particle dropped from rest from a vertical tower of height h. Writing a vector as a column of its components along our choice of axes, this says that the initial conditions are,

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}.$$

Using $\boldsymbol{\omega} \times \mathbf{g} = -\omega g \cos\lambda\, \hat{\mathbf{x}}$, we find that the components, x, y and z of $\mathbf{x}$ are:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \frac{1}{2}gt^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3}\omega g t^3 \cos\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The particle hits the ground when $z = 0$ at $t = \sqrt{2h/g}$. For this t, the x component of the particle's position is

$$\frac{1}{3}\omega \cos\lambda \left(\frac{8h^3}{g}\right)^{1/2}.$$

This says that the particle strikes the ground a little to the East of the base of the tower.

Two views of this are shown in figure 4.5. On the left is the view from a non-inertial frame fixed to the rotating Earth: the particle lands a little to the East of the base of the tower. On the right is a view from an inertial frame, where the Earth and tower are spinning beneath the observer. Now the particle is seen to be projected from the top of the tower. Because the particle is acted upon by the Earth's gravitational attraction, a central force, its angular momentum around the Earth's rotation axis is constant. As the particle falls, it gets closer to the axis, so its angular velocity must increase to keep the angular momentum constant. Therefore, the particle is again seen to get slightly ahead of the tower as it falls.

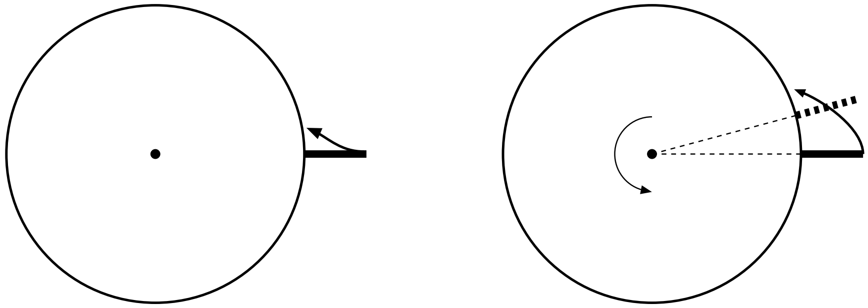

Figure 4.5 Two views of a particle dropped from the top of a tall tower fixed to the rotating Earth. On the left, as seen in a rotating frame fixed to the Earth, and on the right as seen in an inertial frame in which the Earth spins on its axis.

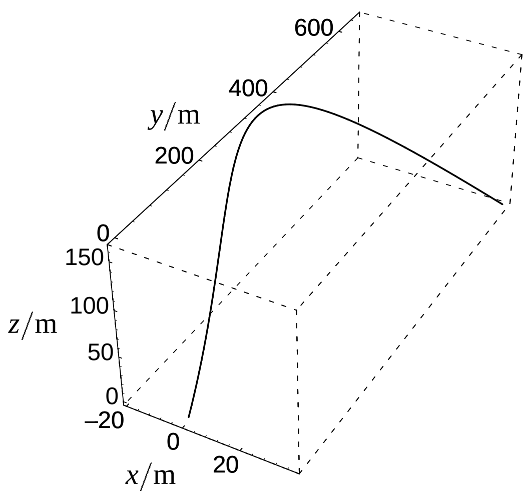

Figure 4.6 Deflection of a cannon shell by Coriolis force when viewed from non-inertial coordinates rotating with the Earth. A shell is fired at elevation angle $\pi/4$ with speed $80\,\mathrm{m\,s^{-1}}$ at latitude 24° in the Northern hemisphere. The Earth's angular velocity is set to $\omega = 0.05\,\mathrm{rad\,s^{-1}}$ to exaggerate the effect.

Shell fired from a cannon A shell is fired due North with speed v from a cannon, with elevation angle $\pi/4$. The initial conditions, taking the origin at the cannon, are now,

$$\mathbf{v} = \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

and the cross product of $\boldsymbol{\omega}$ with the initial velocity is,

$$\boldsymbol{\omega}\times\mathbf{v} = \frac{\omega v}{\sqrt{2}}(\cos\lambda - \sin\lambda)\,\hat{\mathbf{x}}.$$

Substituting in our solution we get:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{vt}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{2}gt^2\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{1}{3}\omega g t^3\cos\lambda\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{\omega v t^2}{\sqrt{2}}(\cos\lambda - \sin\lambda)\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

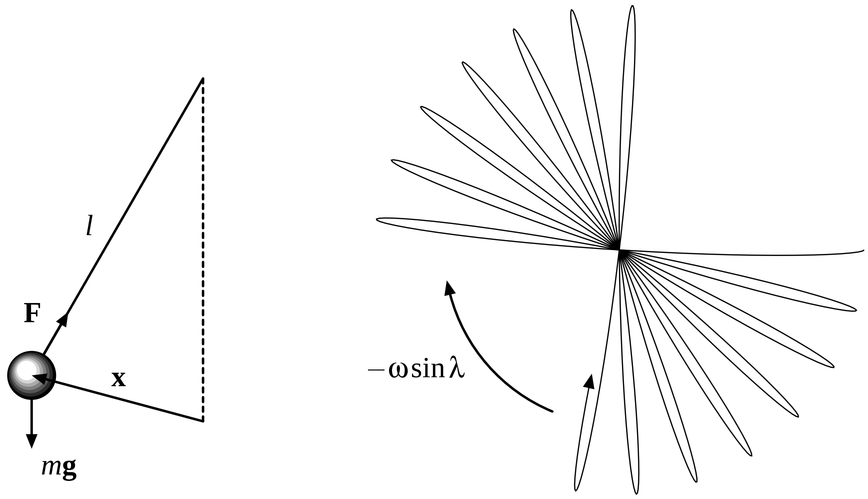

Figure 4.7 Foucault's pendulum and much exaggerated view of the path of the bob. The plane of oscillation rotates with angular velocity $-\omega\sin\lambda$, clockwise when seen from above.

Looking at the z component of this result shows that $z = vt/\sqrt{2} - gt^2/2$, so impact occurs at $t = \sqrt{2}v/g$. The Eastward deflection at impact is then found to be:

$$\frac{\sqrt{2}\omega v^3}{3g^2}\,(3\sin\lambda - \cos\lambda).$$

If $3\sin\lambda > \cos\lambda$ then the deflection at impact will be to the East. This occurs for $\lambda > \tan^{-1}(1/3) = 18.4°$, roughly the latitude of Mexico City or Bombay.

The Eastward deflection is the sum of a positive cubic term in the time t plus a quadratic term in t which is positive for $\lambda > 45°$. So, at Southampton, $\lambda = 51°$, the deflection is Eastward throughout the trajectory, but at latitudes below 45°, the deflection is initially to the West and then changes to the East. Figure 4.6 shows the trajectory up to the impact time, for $\lambda = 24°$, with an initial speed $v = 80\,\mathrm{m\,s^{-1}}$, but using a ridiculously large value, $\omega = 0.05\,\mathrm{rad\,s^{-1}}$, for the Earth's angular velocity to magnify the effect. This value of ω is about 700 times larger than the true value of about $7.3\times10^{-5}\,\mathrm{rad\,s^{-1}}$. If the angular velocity were really as large as $0.05\,\mathrm{rad\,s^{-1}}$, we wouldn't be justified in using our small-ω approximation.

4.3.4 Foucault's Pendulum

If you were to set up a pendulum at the North pole and start it swinging in a plane (as viewed from an inertial frame — one not attached to the Earth), then clearly, according to an observer standing on the Earth, the plane of oscillation would rotate backwards at angular velocity $-\omega$.

At lower latitudes, the phenomenon persists, but gets more and more diluted until it vanishes at the equator. In fact, at latitude λ the plane of oscillation rotates at angular velocity $-\omega\sin\lambda$. This is illustrated, in a very exaggerated fashion, in figure 4.7. At Southampton, latitude 51°, the plane rotates about 10° in one hour. The effect was first demonstrated by Jean Foucault in Paris in 1851[1]. In practice, it is quite hard to start the pendulum with the correct initial conditions: the bob often ends up with a circular or elliptical path where the Foucault rotation is much harder to detect.

[1] For background, see the article *Léon Foucault*, Scientific American (July 1998) pp52–59

We will now derive the result for the rotation of the plane of oscillation. We make our standard choice of coordinates, shown in figure 4.4, with the z-axis along the upward local vertical, $\hat{\mathbf{z}} = -\mathbf{g}^*/g^*$. We will work to first order in the Earth's angular velocity ω, so we will drop the star on g^*. The system we consider is a pendulum of length l, free to swing in any direction with the same period, as illustrated in figure 4.7. The pendulum should be long and heavy so that it will swing for a long time, a matter of hours, in spite of air resistance (which we will neglect).

Measuring the displacement $\mathbf{x}$ of the bob from the bottom of the swing, the equations of motion in our coordinate system are just those of equation (4.1), where $\mathbf{F}$ is the tension in the support cable. In the approximation of small oscillations, we can ignore all z terms compared to x and y. Then, $F_x \approx -mgx/l$ and $F_y \approx -mgy/l$. The x and y equations now become,

$$\begin{aligned} \ddot{x} &= -\omega_0^2 x + 2\omega \sin\lambda \dot{y}, \\ \ddot{y} &= -\omega_0^2 y - 2\omega \sin\lambda \dot{x}, \end{aligned}$$

where we have defined $\omega_0^2 \equiv g/l$, so that ω_0 is the natural angular frequency of the pendulum. To solve these equations, define the complex quantity $\alpha = x + iy$. It is easy to see that the two equations above combine into a single equation for α,

$$\ddot{\alpha} + 2i\omega \sin\lambda \dot{\alpha} + \omega_0^2 \alpha = 0.$$

Look for a solution of the form $\alpha = Ae^{ipt}$. Substituting this form shows that we have a solution provided,

$$\begin{aligned} p &= -\omega \sin\lambda \pm \sqrt{\omega_0^2 + \omega^2 \sin^2\lambda} \\ &\approx -\omega \sin\lambda \pm \omega_0, \end{aligned}$$

where we have used $\omega_0 \gg \omega \sin\lambda$. The general solution is therefore,

$$\alpha = \left(Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right) e^{-i(\omega \sin\lambda)\, t},$$

where A and B are complex constants. With appropriate initial conditions the solution can be given as,

$$\boxed{\alpha = ae^{-i(\omega \sin\lambda)\, t} \cos(\omega_0 t)}.$$

The $\cos(\omega_0 t)$ term describes the usual periodic swing of the pendulum and the $e^{-i(\omega \sin\lambda)\, t}$ term describes the rotation of the plane of oscillation with angular velocity $-\omega \sin\lambda$, as shown in figure 4.7.

Geometric Description* There is a nice geometric way to think about the Foucault Pendulum which allows you to work out the rotation rate without solving a differential equation[2].

Draw parallel lines on a disc and then cut out a segment and fold the remainder into a cone. Choose the disc radius so that the edge of the cone sits on the Earth's surface at latitude λ, with the surface of the cone tangential to the Earth's surface where it touches. Keep the cone fixed in space as the Earth turns beneath it. As the Earth turns, the plane of swing of the Foucault pendulum always remains parallel to the lines drawn on the cone's surface. The construction is shown in figure 4.8. If you

[2] See J B Hart, R E Miller and R L Mills, Am. J. Phys. **55** (1987) 67.

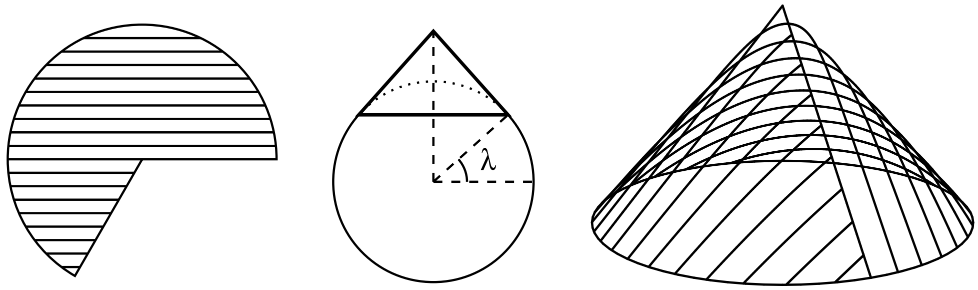

Figure 4.8 A geometric construction to find the rate of rotation of the plane of oscillation of a Foucault pendulum.

think about it, you should be able to figure out the rotation rate from the geometry (try it!).

This is an example of “parallel transport”: the plane of swing of the pendulum is parallel-transported as the Earth rotates. This concept is very important in differential geometry, which underlies general relativity.