Preface

These are notes to accompany the second year core physics course PHYS2006 Classical Mechanics. They are not necessarily complete and are *not* a substitute for the lectures. Certain sections are "starred" (with a star at the end of the section name, like this*): they contain material which is either revision or goes beyond the main line of development of the course. You do not need to consider such sections as part of the syllabus.

Background Information

This course continues the mechanics started in *Energy and Matter*, PHYS1013. It also builds on the study *Motion and Relativity*, PHYS1015. It relates to other physics courses, especially in quantum mechanics and condensed matter. I will try to highlight the importance of identifying symmetries to help with physical understanding. This should come up several times in the course.

In this course we will return to gravity and derive the important result that the gravitational effect of a spherically symmetric object is the same as the effect of a point mass, with the same total mass, at its centre. We then discuss Kepler's laws of planetary motion. This was an early triumph for Newtonian mechanics. To link the observed effects of gravity on the Earth with the force governing celestial motion was a stunning achievement.

We will actually begin, however, by considering the motion of systems of particles, allowing us to study problems such as rocket motion. We will then look at rotational dynamics, applying Newton's Laws to angular motion, encountering angular velocity, angular momentum and, for systems of particles, the moment of inertia. We will see some of the seemingly counterintuitive effects that arise in the motion of spinning objects.

We normally use inertial coordinate systems. However, the rotation of the Earth on its axis makes coordinate systems fixed to the Earth non-inertial. We'll work out the equation of motion in such a reference frame and see the effects that arise, discussing especially the Coriolis term.

Finally, we consider oscillations and waves in systems of coupled oscillators.

Course Information

Prerequisites The course will assume familiarity with the first year physics and mathematics core courses, particularly PHYS1013, PHYS1015, MATH1006/8 and MATH1007.

Teaching Staff Prof. S. Moretti is the course coordinator and principal lecturer. His office is Room 5043 in the School of Physics and Astronomy (building 46) and he can be contacted by email as stefano@soton.ac.uk or by telephone on extension 26829.

Course Structure The course comprises about 30 lectures, three per week. Each week there is a one hour workshop where you work on a problem set. At the workshop you hand in answers for the previous week's problem set and receive marked answers from the problem set handed in the previous week. There are ten workshop sessions.

Class Size and Organisation This is a core course for BSc and MPhys students so all second year physicists attend. There are also some non-physics students. Currently there are about 140+ students in total. For 2014/15 lectures are on Mondays (09:00 to 10:00) and Thursdays (10:00 to 12:00, a double slot with a break in between). There is one problem class (workshop) each week for 11 weeks starting from the first one on Tuesdays (09:00 to 10:00). This year the course is in the second semester.

Course Materials A handout of printed notes is available (a copy is provided for every student at the start of the course). These notes are *not* necessarily complete, however. A copy of the lecturer's own notes is available from the School Office. You may borrow those notes, using a sign-out system. The course has web pages at:

http://www.hep.phys.soton.ac.uk/courses/phys2006/

Study Requirements and Assessment Since it is part of your physics foundation, this course's orientation is towards problem solving, based on a small number of principles. It is very important that you study the weekly problem sheets. They count for 20% of the marks for the course.

The examination will contain two sections, section A with a number of short questions (typically five) all of which must be answered, and section B with four questions from which you must answer two and only two. Section A carries 1/3 and section B carries 2/3 of the examination marks. The way the final mark for this module is worked out is explained in the Student Handbook.

Student Assessment of the Course Informal feedback to the lecturer is always welcomed. Individual problems can usually be dealt with by the workshop leaders, but if several people share a problem they may like to consult the lecturer together. Students' opinions are canvassed by a departmental questionnaire issued around one third of the way through the course. The responses are reviewed by the School's Syllabus Committee and the Staff Student Liaison Committee. There is also a questionnaire at the end of the course, issued by the Faculty of Science.

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Reading List and Syllabus

Reading List

D Acheson, From Calculus to Chaos: an Introduction to Dynamics, Oxford University Press 1997

TL Chow, Classical Mechanics, John Wiley 1995

PA Tipler, Physics for Scientists and Engineers (Vol 1, 5th Edition), Freeman 2004

G R Fowles and G I Cassiday, Analytical Mechanics 5th edition, Saunders College Publishing 1993

A P French, Vibrations and Waves, MIT Introductory Physics Series, Van Nostrand Reinhold 1971

A P French and M G Ebison, Introduction to Classical Mechanics, Van Nostrand Reinhold 1986

T W B Kibble, Classical Mechanics 2nd edition, McGraw-Hill 1973

T W B Kibble and F H Berkshire, Classical Mechanics 5th edition, World Scientific Publishing 2004

J B Marion and S T Thornton, Classical Dynamics of Particles and Systems 4th edition, Saunders College Publishing 1995

Fowles and Cassiday's book is full of examples and is the recommended text, although it stops short of discussing one-dimensional crystal models. The treatment of mechanics in Chow's book parallels the course quite closely and has a modern viewpoint. Kibble or Marion and Thornton cover almost everything, but are mathematically more sophisticated. French and Ebison (and French's book on Vibrations and Waves) have good physical explanations but don't cover all the material.

Acheson's book is recommended as supplementary reading and for general background. Although described by its author as "an introduction to some of the more interesting applications of calculus," this book is principally concerned with dynamics, how things evolve in time, and links quite well to some of the topics in this course.

All others are useful to integrate.

Further, two good foundation books to always have at hand are

K F Riley and M P Hobson, Essential Mathematical Methods for the Physical Sciences, Cambridge University Press, 2011

K F Riley and M P Hobson, Foundation Mathematics for the Physical Sciences, Cambridge University Press, 2011

Syllabus

The numbers of lectures indicated for each section are approximate.

Linear motion of systems of particles [5 lectures]

- centre of mass
- total external force equals rate of change of total momentum (internal forces cancel)
- examples (rocket motion, . . .)

Angular motion [7 lectures]

- rotations, infinitesimal rotations, angular velocity vector
- angular momentum, torque
- angular momentum for a system of particles; internal torques cancel for central internal forces
- rigid bodies, rotation about a fixed axis, moment of inertia, parallel and perpendicular axis theorems, inertia tensor mentioned
- precession (simple treatment: steady precession rate worked out), gyrocompass described

Gravitation and Kepler's Laws [7 lectures]

- law of universal gravitation
- gravitational attraction of spherically symmetric objects
- two-body problem, reduced mass, motion relative to centre of mass
- orbits, Kepler's laws
- energy considerations, effective potential

Non-inertial reference frames [6 lectures]

- fictitious forces
- motion in a frame rotating about a fixed axis, centrifugal and Coriolis terms — apparent gravity, Coriolis deflection, Foucault's pendulum, weather patterns

Normal modes [5 lectures]

- damped and forced harmonic oscillation, resonance (revision)
- coupled oscillators, normal modes
- boundary conditions and eigenfrequencies
- beads on a string

Contents

1

Motion of Systems of Particles

This chapter contains formal arguments showing (i) that the total external force acting on a system of particles is equal to the rate of change of its total linear momentum and (ii) that the total external torque acting is equal to the rate of change of the total angular momentum. Although you should ensure you understand the arguments, the important point is the simple and useful general results which emerge.

1.1 Linear Motion

Consider a system of N particles labelled $1, 2, \ldots, N$ with masses m_i at positions $\mathbf{r}_i$. Let the momentum of the ith particle be $\mathbf{p}_i$. The total force acting and the total linear momentum are

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i \quad \text{and} \quad \mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i,$$

respectively. Summing the equations of motion, $\mathbf{F}_i = \dot{\mathbf{p}}_i$ (Newton's second law), for all the particles immediately leads to

$$\mathbf{F} = \dot{\mathbf{P}}.$$

To make this more useful, we divide up the force $\mathbf{F}_i$ on the ith particle into the external force plus the sum of all the internal forces due to the other particles:

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij}.$$

Here, $\mathbf{F}_{ij}$ is the force on the ith particle due to the jth. The payoff for using this decomposition is that the internal forces are related in pairs by Newton's third law,

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji},$$

and therefore,

$$\mathbf{F} = \sum_{i=1}^{N} \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij} \right) = \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \mathbf{F}_{ij}.$$

The first term on the RHS is simply the total external force, $\mathbf{F}^{\text{ext}}$, and the second term vanishes because the internal forces cancel in pairs. Thus we end up with the result:

$$\boxed{\mathbf{F}^{\text{ext}} = \dot{\mathbf{P}}}. \tag{1.1}$$

- The total external force is equal to the rate of change of the total linear momentum of the system.
- We used Newton's third law to cancel the internal forces in pairs.
- If the external force vanishes, $F^{\text{ext}} = 0$, then $\dot{\mathbf{P}} = 0$, so $\mathbf{P}$ is constant and we can state:

The linear momentum of a system subject to no net external force is conserved.

1.1.1 Centre of Mass

Define the centre of mass, $\mathbf{R}$, by,

$$\mathbf{R} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i,$$

where $M = \sum m_i$ is the total mass.

If the individual masses are constant, then the velocity of the centre of mass is found from,

$$M\dot{\mathbf{R}} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \mathbf{P}.$$

Furthermore, we just saw above that $\mathbf{F}^{\text{ext}} = d\mathbf{P}/dt$. So we have the following results:

$$\boxed{\mathbf{P} = M\dot{\mathbf{R}}} \quad \text{and} \quad \boxed{\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}}. \tag{1.2}$$

- In the *absence* of a net external force, the centre of mass moves with constant velocity. This says (once again) that:

The linear momentum of a system subject to no net external force is conserved.

- If the net external force is non-zero, the centre of mass moves as if the total mass of the system were there, acted on by the total external force.

It is often useful to look at the system of particles with positions measured relative to the centre of mass. If $\boldsymbol{\rho}_i$ is the location of the ith particle with respect to the centre of mass then (see figure 1.1),

$$\boxed{\mathbf{r}_i = \mathbf{R} + \boldsymbol{\rho}_i}. \tag{1.3}$$

1.1.2 Kinetic Energy of a System of Particles

Let's look at the total kinetic energy T of the system using the decomposition in equation (1.3).

$$\begin{aligned} T &= \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_i)^2 \\ &= \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{R}}^2 + \sum_{i=1}^{N} m_i \dot{\boldsymbol{\rho}}_i \cdot \dot{\mathbf{R}} + \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\boldsymbol{\rho}}_i^2. \end{aligned}$$

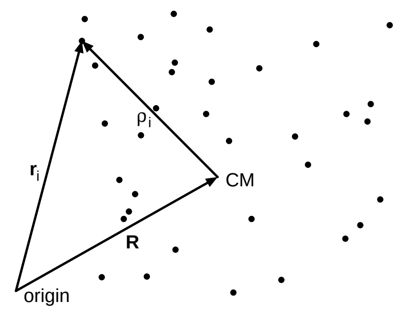


Figure 1.1 Particle positions measured with respect to the Centre of Mass

The second term on the RHS vanishes since $\sum m_i \boldsymbol{\rho}_i = 0$ and $\sum m_i \dot{\boldsymbol{\rho}}_i = 0$ by the definition of the centre of mass. This leaves,

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\boldsymbol{\rho}}_i^2,$$

which we write as,

$$\boxed{T = \frac{1}{2}M\dot{\mathbf{R}}^2 + T_{\mathrm{CM}}}. \tag{1.4}$$

The total kinetic energy has one term from the motion of the centre of mass and a second term from the kinetic energy of motion with respect to the centre of mass. Since particle velocities are different when measured in different inertial reference frames, the kinetic energy will in general be different in different frames. However, T_{CM}, the kinetic energy with respect to the center of mass is the *same* in all inertial frames and is an "internal" kinetic energy of the system (the sum of T_{CM} and the potential energy due to the internal interactions is the total internal energy, U, as used in thermodynamics). To prove this, note that a Galilean transformation from a frame S to a frame S' moving at velocity $\mathbf{v}$ with respect to S changes particle positions by:

$$\mathbf{r}_i \to \mathbf{r}_i' = \mathbf{r}_i - \mathbf{v}t.$$

The centre of mass transforms similarly,

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} \to \mathbf{R}' = \frac{\sum m_i \mathbf{r}_i'}{\sum m_i} = \mathbf{R} - \mathbf{v}t,$$

so that positions and velocities with respect to the centre of mass are *unchanged*:

$$\begin{array}{ccccccccc} \boldsymbol{\rho}_i' & = & \mathbf{r}_i' - \mathbf{R}' & = & (\mathbf{r}_i - \mathbf{v}t) - (\mathbf{R} - \mathbf{v}t) & = & \mathbf{r}_i - \mathbf{R} & = & \boldsymbol{\rho}_i \\ \dot{\boldsymbol{\rho}}_i' & = & \dot{\mathbf{r}}_i' - \dot{\mathbf{R}}' & = & (\dot{\mathbf{r}}_i - \mathbf{v}) - (\dot{\mathbf{R}} - \mathbf{v}) & = & \dot{\mathbf{r}}_i - \dot{\mathbf{R}} & = & \dot{\boldsymbol{\rho}}_i \end{array}$$

The decomposition of the kinetic energy in equation (1.4) can be useful in problem solving. For example, if a ball rolls down a ramp, you can express the kinetic energy as a sum of one term coming from the linear motion of the centre of mass plus another term for the rotational motion about the centre of mass (the kinetic energy of rotational motion is discussed further later in the notes).

System of Two Particles Now apply the kinetic energy expression in equation (1.4) to a system of two particles. Write the particle velocities as $\mathbf{u}_1 = \dot{\mathbf{r}}_1$ and $\mathbf{u}_2 = \dot{\mathbf{r}}_2$, so that:

$$\mathbf{u}_1 = \dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_1 \qquad \text{and} \qquad \mathbf{u}_2 = \dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_2.$$

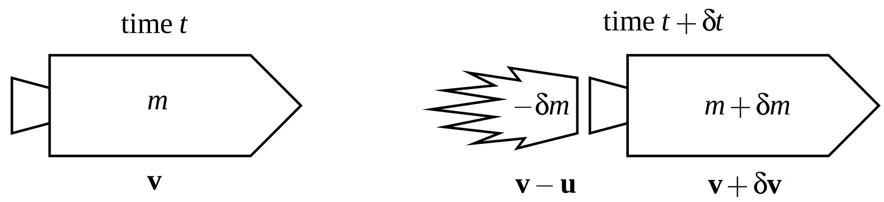


Figure 1.2 Motion of a rocket. We consider the rocket at two closely spaced instants of time, t and $t+\delta t$.

Subtracting these two equations gives $\mathbf{u}_1 - \mathbf{u}_2 = \dot{\boldsymbol{\rho}}_1 - \dot{\boldsymbol{\rho}}_2$, while the centre of mass condition states that $m_1\dot{\boldsymbol{\rho}}_1 + m_2\dot{\boldsymbol{\rho}}_2 = 0$. We can thus solve for $\dot{\boldsymbol{\rho}}_1$ and $\dot{\boldsymbol{\rho}}_2$:

$$\dot{\boldsymbol{\rho}}_1 = \frac{m_2(\mathbf{u}_1 - \mathbf{u}_2)}{m_1 + m_2}, \qquad \dot{\boldsymbol{\rho}}_2 = \frac{-m_1(\mathbf{u}_1 - \mathbf{u}_2)}{m_1 + m_2}.$$

Substituting these in the kinetic energy expression gives,

$$T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\,\frac{m_1 m_2}{m_1 + m_2}\,(\mathbf{u}_1 - \mathbf{u}_2)^2.$$

The quantity $m_1 m_2/(m_1 + m_2)$ appearing here is called the *reduced mass*. We will meet it again (briefly) in chapter 3 on Kepler's laws.

1.1.3 Examples

Rocket Motion We can use our results for the motion of a system of particles to describe so-called "variable mass" problems, where the mass of the (part of) the system we are interested in changes with time. A prototypical example is the motion of a rocket in deep space. The rocket burns fuel and ejects the combustion products at high speed (relative to the rocket), thereby propelling itself forward. To describe this quantitatively, we refer to the diagram in figure 1.2 and proceed as follows.

We consider the rocket at two closely spaced instants of time. At time t the rocket and its remaining fuel have mass m and velocity $\mathbf{v}$. In a short additional interval δt the rocket's mass changes to $m + \delta m$ as it burns a mass $-\delta m$ of fuel (note that δm is *negative* since the rocket uses up fuel for propulsion) and the rocket's velocity changes to $\mathbf{v} + \delta\mathbf{v}$. The exhaust gases are ejected with velocity $-\mathbf{u}$ with respect to the rocket, which is velocity $\mathbf{v} - \mathbf{u}$ with respect to an external observer. Hence, at time $t + \delta t$ we have a rocket of mass $m + \delta m$ moving with velocity $\mathbf{v} + \delta\mathbf{v}$ together with a mass $-\delta m$ of gas with velocity $\mathbf{v} - \mathbf{u}$.

If the rocket is in deep space, far from any stars or planets, there is no gravitational force or other external force on the system, so its overall linear momentum is conserved. Therefore, we may equate the linear momentum of the system at times t and $t + \delta t$,

$$m\mathbf{v} = (m + \delta m)(\mathbf{v} + \delta\mathbf{v}) - \delta m(\mathbf{v} - \mathbf{u}).$$

Cancelling terms we find,

$$\mathbf{u}\delta m + m\delta\mathbf{v} + \delta m\delta\mathbf{v} = 0.$$

We take the limit $\delta t \to 0$, so that the $\delta m\delta\mathbf{v}$ term, which is second order in infinitesimal quantities, drops out, leaving:

$$\mathbf{u}\frac{dm}{m} = -d\mathbf{v}.$$

If the rocket initially has velocity $\mathbf{v}_i$ when its mass is m_i, and ends up with velocity $\mathbf{v}_f$ when its mass is m_f, we integrate this equation to find:

$$\boxed{\mathbf{v}_f = \mathbf{v}_i + \mathbf{u}\ln\left(\frac{m_i}{m_f}\right)}. \tag{1.5}$$

The fact that the increase in the rocket's speed depends logarithmically on the ratio of initial and final masses is the reason why rockets are almost entirely made up of fuel when they are launched (the function $\ln x$ grows *very* slowly with x). It also explains why multi-stage rockets are advantageous: once you have burnt up some fuel, you don't want to carry around the structure that contained it, since this will reduce the ratio m_i/m_f for the subsequent motion.

Rope Falling Onto a Table Here we'll consider a system where an external force acts. A flexible rope with mass per unit length ρ is suspended just above a table. The rope is released from rest. Find the force on the table when a length x of the rope has fallen to the table.

Our system here is the rope. The external forces in the vertical direction are the weight of the rope, ρag, acting downwards plus an upward normal force F exerted on the rope by the tabletop. We want to determine F.

The rope falls freely onto the table, so its downward acceleration is g. If we let $v = \dot{x}$, this means that $\dot{v} = g$ and $v^2 = 2gx$.

Suppose that a length x of the rope has reached the table top after time t, when the speed of the falling section is v. A short time δt later, the length of rope on the table is $x + \delta x$ and the speed of the falling section is $v + \delta v$. The downward components of the system's total momentum at times t and $t + \delta t$ are therefore:

$$\begin{aligned} p(t) &= \rho(a-x)v, \\ p(t+\delta t) &= \rho(a-x-\delta x)(v+\delta v). \end{aligned}$$

Working to first order in small quantities,

$$\delta p = p(t+\delta t) - p(t) = \rho(a-x)\delta v - \rho v \delta x.$$

Taking the limit $\delta t \to 0$, we find that the rate of change of momentum is,

$$\frac{dp}{dt} = \rho(a-x)\dot{v} - \rho v\dot{x} = \rho(a-x)g - 2\rho xg.$$

Therefore, equating the external force to the rate of change of momentum gives,

$$\rho ag - F = \rho(a-x)g - 2\rho xg,$$

or finally,

$$F = 3\rho xg.$$

1.2 Angular Motion

The angular equation of motion for each particle is

$$\mathbf{r}_i \times \mathbf{F}_i = \frac{d}{dt}(\mathbf{r}_i \times \mathbf{p}_i).$$

The total angular momentum of the system and the total torque acting are:

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \quad \text{and} \quad \boldsymbol{\tau} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i$$

As before we split the total force on each particle into external and internal parts. We then make a corresponding split in the total torque:

$$\begin{aligned} \boldsymbol{\tau} &= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_i \times \sum_{j \neq i} \mathbf{F}_{ij} \\ &\equiv \boldsymbol{\tau}^{\text{ext}} + \boldsymbol{\tau}^{\text{int}}. \end{aligned}$$

Recall that in the linear case, we were able to cancel the internal forces in pairs, because they satisfied Newton's third law. What is the corresponding result here? In other words, when can we ignore $\boldsymbol{\tau}^{\text{int}}$? To answer this, decompose $\boldsymbol{\tau}^{\text{int}}$ as follows,

$$\begin{aligned} \boldsymbol{\tau}^{\text{int}} &= \mathbf{r}_1 \times (\mathbf{F}_{12} + \mathbf{F}_{13} + \cdots + \mathbf{F}_{1N}) \\ &\quad + \mathbf{r}_2 \times (\mathbf{F}_{21} + \mathbf{F}_{23} + \cdots + \mathbf{F}_{2N}) + \cdots \\ &= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12} + (\text{other pairs}). \end{aligned}$$

We have used Newton's third law to obtain the last line.

Now, if the internal forces act along the lines joining the particle pairs, then all the terms $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}$ vanish and $\boldsymbol{\tau}^{\text{int}} = 0$. Thus $\boldsymbol{\tau}^{\text{int}} = 0$ for *central* internal forces. Examples are gravity and the Coulomb force.

With this proviso we obtain the result,

$$\sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \frac{d}{dt} \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i,$$

which is rewritten as,

$$\boxed{\boldsymbol{\tau}^{\text{ext}} = \dot{\mathbf{L}}}.$$

- This result applies when we use coordinates in an inertial frame (one in which Newton's laws apply).
- Note that we used both Newton's third law and the condition that the forces between particles were central in order to reach our result.

1.2.1 Angular Motion About the Centre of Mass

We will now see that taking moments about the centre of mass also leads to a simple result. To do this, look at the total angular momentum using the centre of mass coordinates:

$$\begin{aligned} \mathbf{L} &= \sum_{i=1}^{N} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i = \sum_{i=1}^{N} (\mathbf{R} + \boldsymbol{\rho}_i) \times m_i (\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_i) \\ &= \sum_{i=1}^{N} \mathbf{R} \times m_i \dot{\mathbf{R}} + \sum_{i=1}^{N} \mathbf{R} \times m_i \dot{\boldsymbol{\rho}}_i + \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\mathbf{R}} + \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i. \end{aligned}$$

The second and third terms on the RHS vanish since $\sum m_i \boldsymbol{\rho}_i = 0$ and $\sum m_i \dot{\boldsymbol{\rho}}_i = 0$ by the definition of the centre of mass. This leaves,

$$\mathbf{L} = \mathbf{R} \times M\dot{\mathbf{R}} + \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i,$$

which we write as,

$$\boxed{\mathbf{L} = \mathbf{R} \times M\dot{\mathbf{R}} + \mathbf{L}_{\mathrm{CM}}}. \tag{1.6}$$

The total angular momentum therefore has two terms, which can be interpreted as follows. The first arises from the motion of the centre of mass about the origin of coordinates: this is called the *orbital* angular momentum and takes different values in different inertial frames. The second term, $\mathbf{L}_{\mathrm{CM}}$, arises from the angular motion about (relative to) the centre of mass (think of the example of a spinning planet orbiting the Sun): this is the *same* in all inertial frames and is an *intrinsic* or *spin* angular momentum (the proof of this is like the one given for T_{CM}, the kinetic energy relative to the CM, below equation (1.4) on page 3).

Finally, we take the time derivative of the last equation to obtain,

$$\begin{aligned}
\frac{d\mathbf{L}_{\mathrm{CM}}}{dt} &= \frac{d\mathbf{L}}{dt} - \mathbf{R} \times M\ddot{\mathbf{R}} \quad = \quad \boldsymbol{\tau}^{\mathrm{ext}} - \mathbf{R} \times \mathbf{F}^{\mathrm{ext}} \\
&= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\mathrm{ext}} - \sum_{i=1}^{N} \mathbf{R} \times \mathbf{F}_i^{\mathrm{ext}} \\
&= \sum_{i=1}^{N} (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i^{\mathrm{ext}} \\
&= \sum_{i=1}^{N} \boldsymbol{\rho}_i \times \mathbf{F}_i^{\mathrm{ext}} \quad \equiv \quad \boldsymbol{\tau}_{\mathrm{CM}}^{\mathrm{ext}}.
\end{aligned}$$

So we've found two results we can use when considering torques applied to a system:

$$\boxed{\boldsymbol{\tau}^{\mathrm{ext}} = \dot{\mathbf{L}}} \qquad \text{and} \qquad \boxed{\boldsymbol{\tau}_{\mathrm{CM}}^{\mathrm{ext}} = \dot{\mathbf{L}}_{\mathrm{CM}}}. \tag{1.7}$$

- These two equations say you can take moments either about the origin of an inertial frame, or about the centre of mass (even if the centre of mass is itself accelerating).

- Furthermore, in either case:

The angular momentum of a system subject to no external torque is constant.

1.3 Commentary

In deriving the general results above we assumed the validity of Newton's third law, so that we could cancel internal forces in pairs. We also assumed that the forces were central so that we could cancel internal torques in pairs. The assumption of central internal forces is very strong and we know of examples, such as the electromagnetic forces between moving particles, which are *not* central.

All we actually require is the validity of the results in equations (1.1) and (1.7). It is perhaps better to regard them as basic assumptions whose justification is that their consequences agree with experiment.

For the puzzle associated with the electromagnetic forces mentioned above, the resolution is that you have to ascribe energy, momentum and angular momentum to the electromagnetic field itself.

2

Rotational Motion of Rigid Bodies

2.1 Rotations and Angular Velocity

A rotation $R(\hat{\mathbf{n}}, \theta)$ is specified by an axis of rotation, defined by a unit vector $\hat{\mathbf{n}}$ (2 parameters) and an angle of rotation θ (one parameter). Since you have a direction and a magnitude, you might suspect that rotations could be represented in some way by vectors. However, rotations through finite angles are *not* vectors, because they do not commute when you "add" or combine them by performing different rotations in succession. This is illustrated in figure 2.1

Infinitesimal rotations *do* commute when you combine them, however. To see this, consider a vector $\mathbf{A}$ which is rotated through an infinitesimal angle $d\phi$ about an axis $\hat{\mathbf{n}}$, as shown in figure 2.2. The change, $d\mathbf{A}$ in $\mathbf{A}$ under this rotation is a tiny vector from the tip of $\mathbf{A}$ to the tip of $\mathbf{A} + d\mathbf{A}$. The figure illustrates that $d\mathbf{A}$ is perpendicular to both $\mathbf{A}$ and $\hat{\mathbf{n}}$. Moreover, if $\mathbf{A}$ makes an angle θ with the axis $\hat{\mathbf{n}}$, then, in magnitude, $|d\mathbf{A}| = A \sin\theta \, d\phi$, so that as a vector equation,

$$d\mathbf{A} = \hat{\mathbf{n}} \times \mathbf{A} \, d\phi.$$

This has the right direction and magnitude.

If you perform a second infinitesimal rotation, then the change will be some new $d\mathbf{A}'$ say. The total change in $\mathbf{A}$ is then $d\mathbf{A} + d\mathbf{A}'$, but since addition of vectors

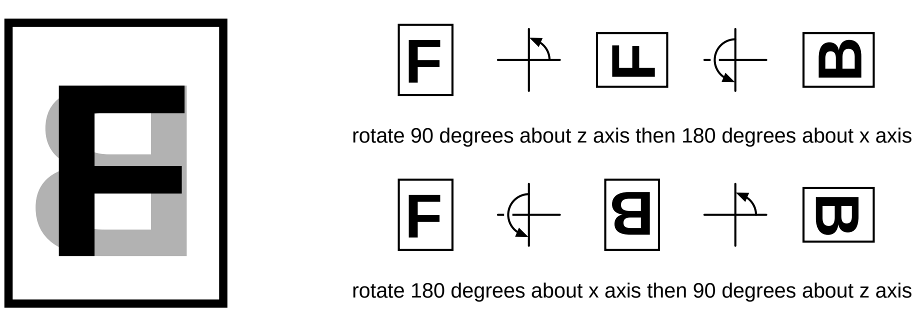


Figure 2.1 Finite rotations do not commute. A sheet of paper has the letter "F" on the front and "B" on the back (shown light grey in the figure). Doing two finite rotations in different orders produces a different final result.

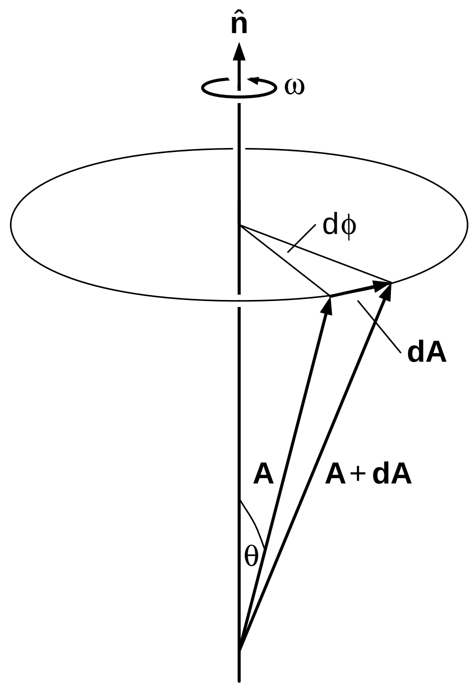


Figure 2.2 A vector is rotated through an infinitesimal angle about an axis.

commutes, this is the same as $d\mathbf{A}' + d\mathbf{A}$. So, infinitesimal rotations *do* combine as vectors.

Now think of $\mathbf{A}$ as denoting a position vector, rotating around the axis with angular velocity $d\phi/dt = \dot{\phi}$, with the length of $\mathbf{A}$ fixed. This describes a particle rotating in a circle about the axis. The velocity of the particle is,

$$\mathbf{v} = \frac{d\mathbf{A}}{dt} = \hat{\mathbf{n}} \times \mathbf{A}\dot{\phi}.$$

We can define the vector *angular velocity*,

$$\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{n}},$$

and then,

$$\boxed{\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}}. \tag{2.1}$$

It's not necessary to think of $\mathbf{A}$ as a position vector, so this result describes the rate of change of any rotating vector of fixed length.

2.2 Moment of Inertia

We will consider the rotational motion of *rigid bodies*, where the relative positions of *all* the particles in the system are fixed. Specifying how one point in the body moves around an axis is then sufficient to specify how the whole body moves. The idea of a rigid body is clearly an idealisation. Real bodies are not rigid and will deform, however slightly, when subject to loads. Their constituents are also subject to random thermal motion. Nonetheless there are many situations where the deformation and any thermal motion can be ignored.

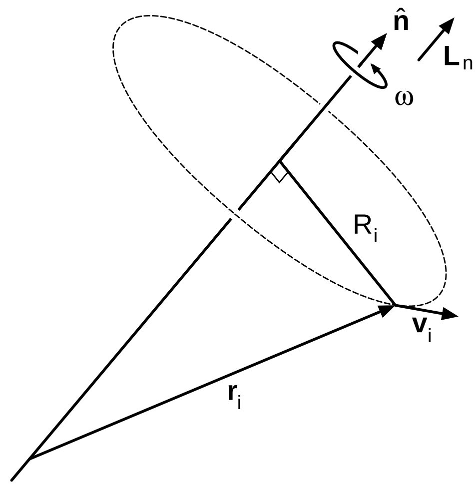


Figure 2.3 Rigid body rotation about a fixed axis.

The general motion of a rigid body with a moving rotation axis is complicated, so we will specialise to a *fixed* axis at first. We can extend our analysis to *laminar* motion, where the axis can move, without changing its direction: an example is given by a cylinder rolling in a straight line down an inclined plane. We will later discuss precession, where the axis itself rotates.

For a rigid body rotating about a fixed axis, what property controls the angular acceleration produced by an external torque? The property will be the rotational analogue of mass (which tells you the linear acceleration produced by a given force). It is known as the *moment of inertia*, sometimes abbreviated (in these notes anyway) as *MoI*.

To find out how to define the MoI, look at the kinetic energy of rotation. Let $\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$, so that $\hat{\mathbf{n}}$ specifies the rotation axis. Let m_i be the mass of the ith particle in the body and let R_i be the perpendicular distance of the ith particle from the rotation axis. The geometry is illustrated in figure 2.3. Since the body is rigid, R_i is a fixed distance for each i and ω is the same for all particles in the body. The kinetic energy is

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i R_i^2 \omega^2 = \frac{1}{2} I \omega^2,$$

where the last equality allows us to define the MoI about the given axis, according to,

$$\boxed{I \equiv \sum_i m_i R_i^2}.$$

The contribution of an element of mass to I grows quadratically with its distance from the rotation axis. Note the analogy between $\frac{1}{2}mv^2$ for the kinetic energy of a particle moving with speed v and $\frac{1}{2}I\omega^2$ for the kinetic energy of a body with moment of inertia I rotating with angular speed ω.

If the position vector $\mathbf{r}_i$ of the ith particle is measured from a point on the rotation axis, then $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$ and $v_i = |\boldsymbol{\omega} \times \mathbf{r}_i| = R_i\omega$. This is an application of the result in equation (2.1) for the rate of change of a rotating vector.

The moment of inertia is one measure of the mass distribution of an object. Other characteristics of the mass distribution we have already met are the total mass and the location of the centre of mass.

For a continuous mass distribution, simply replace the sums over discrete particles with integrals over the mass distribution,

$$\boxed{I = \int_{\text{body}} R^2\, dm = \int_{\text{body}} R^2 \rho\, d^3\mathbf{r}}.$$

Here, $dm = \rho\, d^3\mathbf{r}$ is a mass element, ρ is the mass density and $d^3\mathbf{r}$ is a volume element.

It is sometimes convenient to use the *radius of gyration*, k, defined by

$$\boxed{I \equiv Mk^2}.$$

A single particle of mass equal to the total mass of the body at distance k from the rotation axis will have the same moment of inertia as the body.

Now look at the component L_n in the direction of the rotation axis of the (vector) angular momentum about some point on the axis (see figure 2.3). This is obtained by summing all the contributions of momenta perpendicular to the axis times the perpendicular separation from the axis,

$$\boxed{L_n = \sum_i R_i (m_i R_i \omega) = I\omega}.$$

The subscript n labels the rotation axis. Note that the angular momentum of the ith particle is $\mathbf{L}_i = \mathbf{r}_i \times m_i \mathbf{v}_i$, and the component of this in the direction of $\hat{\mathbf{n}}$ is,

$$\hat{\mathbf{n}} \cdot (\mathbf{r}_i \times m_i \mathbf{v}_i) = \hat{\mathbf{n}} \cdot (\mathbf{r}_i \times m_i (\boldsymbol{\omega} \times \mathbf{r}_i)) = m_i R_i^2 \omega,$$

which is just what appears in the sum giving L_n.

If $\hat{\mathbf{n}}$ is a symmetry axis then L_n is the only non-zero component of the total angular momentum $\mathbf{L}$. However, in general, $\mathbf{L}$ need not lie along the axis, or equivalently, $\mathbf{L}$ *need not be parallel to* $\boldsymbol{\omega}$.

Taking components of the angular equation of motion, $\boldsymbol{\tau} = d\mathbf{L}/dt$ along the axis gives,

$$\tau_n = \frac{dL_n}{dt} = I\dot{\omega} = I\ddot{\phi},$$

if ϕ measures the angle through which the body has rotated from some reference position.

2.3 Two Theorems on Moments of Inertia

2.3.1 Parallel Axis Theorem

$$\begin{aligned} I_{\text{CM}} &= \text{Moment of Inertia (MoI) about axis through centre of mass (CM)} \\ I &= \text{MoI about parallel axis at distance } d \text{ from axis through CM} \end{aligned}$$

The parallel axis theorem states:

$$\boxed{I = I_{\text{CM}} + Md^2},$$

where M is the total mass. To prove this result, choose coordinates with the z-axis along the direction of the two parallel axes, as shown in figure 2.4. Then,

$$I = \sum_{i=1}^{N} m_i (x_i^2 + y_i^2).$$

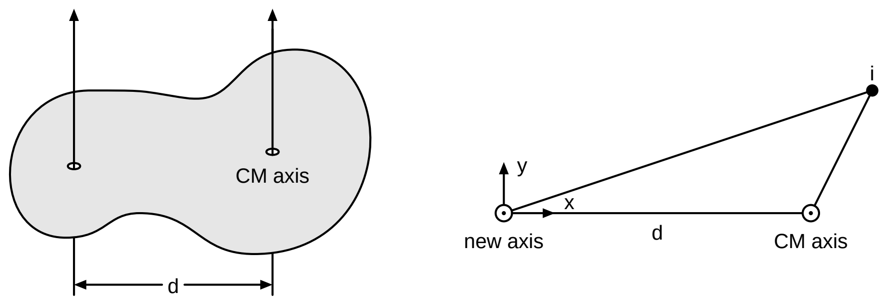


Figure 2.4 Parallel axis theorem. In the right hand figure, we are looking vertically down in the z direction.

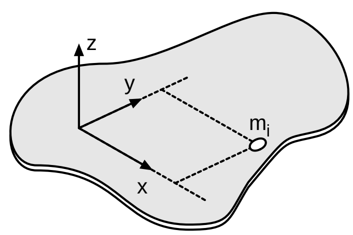


Figure 2.5 Perpendicular axis theorem for thin flat plates.

We can also choose the x-direction to run from the new axis to the CM axis. Then,

$$x_i = d + \rho_{ix} \quad \text{and} \quad y_i = \rho_{iy}$$

where ρ_{ix} and ρ_{iy} are coordinates with respect to the CM. The expression for I becomes:

$$I = \sum_{i=1}^{N} m_i((d+\rho_{ix})^2 + \rho_{iy}^2) = \sum_{i=1}^{N} m_i(\rho_{ix}^2 + \rho_{iy}^2 + d^2 + 2d\rho_{ix}).$$

The last term above contains $\sum m_i \rho_{ix}$ which vanishes by the definition of the CM. The remaining terms give I_{CM} and Md^2 and the result is proved.

2.3.2 Perpendicular Axis Theorem

This applies for thin flat plates of arbitrary shapes, which we take to lie in the x-y plane, as shown in figure 2.5. Let I_x, I_y and I_z be the MoI about the x, y and z axes respectively. The perpendicular axis theorem states:

$$\boxed{I_z = I_x + I_y}.$$

The proof of this is very quick. Just observe that since we have a thin flat plate, then

$$I_x = \sum_{i=1}^{N} m_i y_i^2 \quad \text{and} \quad I_y = \sum_{i=1}^{N} m_i x_i^2.$$

But

$$I_z = \sum_{i=1}^{N} m_i(x_i^2 + y_i^2),$$

and the result is immediate.

In both these results we have assumed discrete distributions of point masses. For continuous mass distributions, simply replace the sums by integrations. For example,

$$I_z = \sum_{i=1}^{N} m_i(x_i^2 + y_i^2) \longrightarrow \int (x^2 + y^2)\, dm.$$

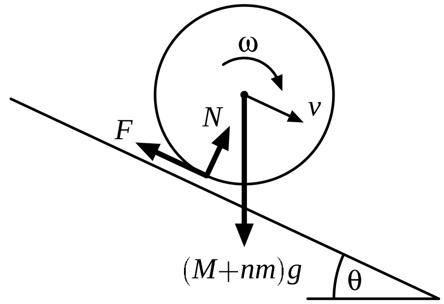


Figure 2.6 Wheel rolling down a slope.

2.4 Examples

Moment of Inertia of a Thin Rod Find the moment of inertia of a uniform thin rod of length $2a$ about an axis perpendicular to the rod through its centre of mass. Also find the moment of inertia about a parallel axis through the end of the rod.

Let ρ be the mass per unit length of the rod and let x measure position along the rod starting from the centre of mass (so $-a \leq x \leq a$). For an element of the rod of length dx the mass is ρdx and the moment of inertia of the element is $\rho x^2\,dx$. Therefore the total moment of inertia is given by the integral:

$$I_{\mathrm{CM}} = \int_{-a}^{a} \rho x^2\,dx = \frac{2}{3}\rho a^3.$$

The total mass is $m = 2\rho a$, and therefore,

$$I_{\mathrm{CM}} = \frac{1}{3} m a^2.$$

Applying the parallel axis theorem, the moment of inertia about one end of the rod is,

$$I_{\mathrm{end}} = I_{\mathrm{CM}} + m a^2 = \frac{4}{3} m a^2.$$

Spoked Wheel A wheel of radius a comprises a thin rim of mass M and n spokes, each of mass m, which may be considered as thin rods terminating at the centre of the wheel. If the wheel rolls without slipping down a plane inclined at angle θ to the horizontal, as depicted in figure 2.6, what is the linear acceleration of its centre of mass?

We will apply the angular equation of motion about the centre of mass (see equation (1.7) on page 7), and the linear equation of motion (see equation (1.2) on page 2) in a direction parallel to the sloping plane. If the angular velocity of the wheel is ω, then the no-slip condition says that its speed is $v = a\omega$. Choose directions so that ω and v are both positive when the wheel rolls downhill.

The angular equation of motion applied to the wheel about its centre of mass says $\tau^{\mathrm{ext}}_{\mathrm{CM}} = I_{\mathrm{CM}}\,\dot{\omega}$. The external torque comes from the frictional force F acting up the sloping plane at the point of contact with the wheel. Using the result above for the MoI of a rod (remembering that the rod length is now a instead of $2a$), we find,

$$I_{\mathrm{CM}} = M a^2 + \frac{n}{3} m a^2.$$

The angular equation of motion then gives,

$$F a = (M a^2 + \frac{n}{3} m a^2)\,\dot{\omega}.$$

The component of the linear equation of motion in a direction down the plane gives,

$$-F+(M+nm)g\sin\theta=(M+nm)a\dot{\omega}.$$

We now eliminate F and solve for $a\dot{\omega}$, which gives the linear acceleration as,

$$a\dot{\omega}=\frac{3(M+nm)g\sin\theta}{6M+4nm}.$$

Alternatively, since the normal reaction (N in figure 2.6) and frictional forces on the wheel do no work, we can apply the conservation of the kinetic plus (gravitational) potential energy. Applying our result in equation (1.4) on page 3 for the kinetic energy of a system, we find:

$$\frac{1}{2}(M+nm)v^2+\frac{1}{2}I_{\mathrm{CM}}\omega^2-(M+nm)gx\sin\theta=\mathrm{const},$$

where x is the distance moved starting from some reference point. Using $v=\dot{x}=a\omega$ and differentiating with respect to time gives

$$\frac{1}{3}(6M+4nm)\dot{x}\ddot{x}=(M+nm)g\sin\theta\,\dot{x},$$

which leads to the same result as before for the acceleration $a\dot{\omega}=\ddot{x}$.

2.5 Precession

Spinning bodies tend to *precess* under the action of a gravitational torque. We'll work out the steady precession rate for a spinning top. Figure 2.7 shows a top supported at a fixed pivot point. We will apply the angular equation of motion $\boldsymbol{\tau}=d\mathbf{L}/dt$ about the pivot. As drawn, the torque about the pivot due to the weight of the top points into the paper. Hence, the angular momentum $\mathbf{L}$ of the top must change by moving into the paper. If the top is spinning very fast about its axis, then $\mathbf{L}$ is, to a very good approximation, aligned with the top's axis. So, the top will tend to turn bodily, or *precess* around a vertical axis. It may help to think of the torque $\boldsymbol{\tau}$ pushing the tip of $\mathbf{L}$ around.

We can calculate the precession frequency quite easily. Assume that $\mathbf{L}$ is large so that the total angular momentum of the top is given entirely by the spin, and ignore any contribution due to the slow precession of the top about the vertical axis. The torque is given by,

$$\boldsymbol{\tau}=\mathbf{r}\times\mathbf{F},$$

where $\mathbf{r}$ is the vector from the pivot to the top's centre of mass and $\mathbf{F}=m\mathbf{g}$ is the top's weight. In magnitude,

$$\tau=mgr\sin\alpha,$$

where the top's axis makes an angle α with the vertical.

If the top precesses through an infinitesimal angle $d\phi$ about the vertical axis, then the magnitude of the change in $\mathbf{L}$ is,

$$dL=Ld\phi\sin\alpha.$$

If $\dot{\phi}=\omega_{\mathrm{p}}$ is the precession angular velocity, then,

$$\frac{dL}{dt}=L\omega_{\mathrm{p}}\sin\alpha.$$

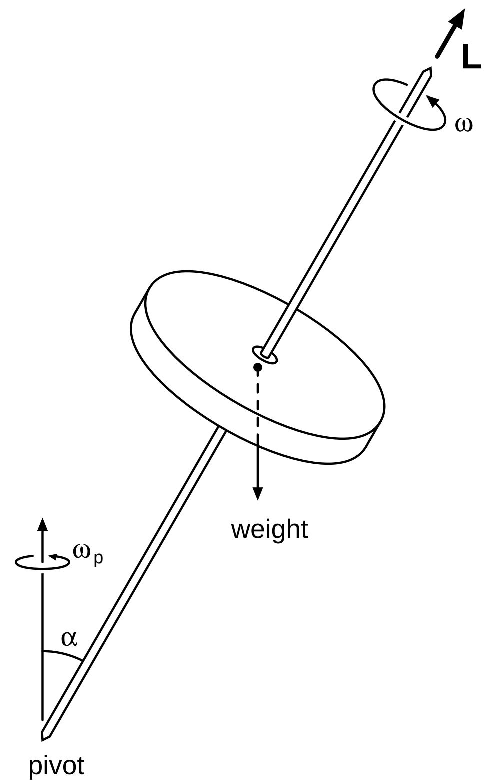


Figure 2.7 A spinning top will precess under gravity.

Applying the equation of motion, taking the magnitude of both sides, gives:

$$mgr \sin\alpha = L\omega_\mathrm{p} \sin\alpha.$$

The $\sin\alpha$ terms cancel and the final answer comes out independent of the angle which the top makes with the vertical. The precession angular velocity is given by,

$$\boxed{\omega_\mathrm{p} = \frac{mgr}{L}}.$$

A full treatment of the motion of a top is complicated. Steady precession is a special motion: in general the top tends to nod up and down, or nutate, as it precesses.

2.6 Gyroscopic Navigation

A gyrocompass is a spinning top mounted in a frame so that its axis is constrained to be horizontal with respect to the Earth, see figure 2.8. As the Earth turns, the axis turns with it, causing the end of the axis labelled A in the figure to be raised upwards and the end B to be pushed down (as seen from a fixed frame not attached to the Earth). This means that there is a torque on the gyroscope which is perpendicular to the spin angular momentum **L** and points between the North and West when the compass is oriented as in the figure.

From the angular equation of motion, $\boldsymbol{\tau} = d\mathbf{L}/dt$, this torque will tend to push **L** towards the North. If **L** points between North and West, the torque again tries to line up **L** with the North-South axis. The gyrocompass will thus tend to oscillate with

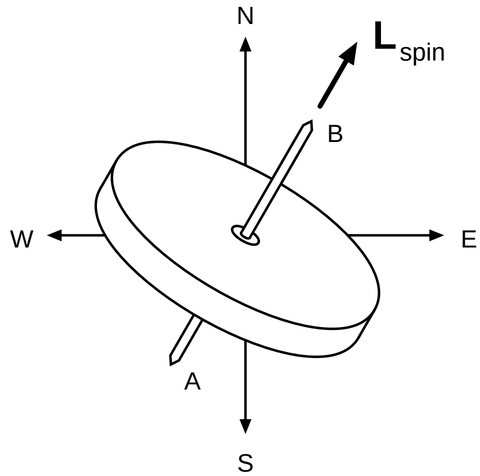


Figure 2.8 A gyrocompass.

its spin direction oscillating about the N-S axis. If you apply some damping, then it will tend to settle down with its spin along the N-S line.

2.7 Inertia Tensor*

Now let's look at the moment of inertia in more detail. So far when we've considered the MoI for a body rotating around a fixed axis, we've always looked at the component L_n of the angular momentum $\mathbf{L}$ along the direction of the axis $\hat{\mathbf{n}}$. Now let's look at *all* the components of $\mathbf{L}$. From the definition of angular momentum we have,

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i = \sum_{i=1}^{N} \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) = \sum_{i=1}^{N} m_i(\mathbf{r}_i \cdot \mathbf{r}_i\,\boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathbf{r}_i\,\mathbf{r}_i),$$

where we have used $\mathbf{p}_i = m_i\mathbf{v}_i = m_i\boldsymbol{\omega} \times \mathbf{r}_i$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$. We also applied a standard result for the vector triple product, $\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{r}_i \cdot \mathbf{r}_i \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathbf{r}_i \mathbf{r}_i$. Rewrite this as a matrix equation giving the components of $\mathbf{L}$ in terms of the components of $\boldsymbol{\omega}$ (the summations run over $i = 1, \ldots, N$):

$$\begin{aligned}
\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} &= \begin{pmatrix} \sum m_i(y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i y_i x_i & \sum m_i(z_i^2 + x_i^2) & -\sum m_i y_i z_i \\ -\sum m_i z_i x_i & -\sum m_i z_i y_i & \sum m_i(x_i^2 + y_i^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \\
&= \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.
\end{aligned}$$

This is given more succinctly as,

$$\mathbf{L} = \mathbf{I}\,\boldsymbol{\omega},$$

where $\mathbf{I}$ is the matrix, known as the *inertia tensor* which acts on $\boldsymbol{\omega}$ to give $\mathbf{L}$. Remembering that $\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$, our old results are recovered from,

$$T = \frac{1}{2}\hat{\mathbf{n}}^T \mathbf{I}\,\hat{\mathbf{n}}\,\omega^2 \qquad \text{and} \qquad L_n = \hat{\mathbf{n}}^T \mathbf{I}\,\hat{\mathbf{n}}\,\omega,$$

so we can define

$$I_n \equiv \hat{\mathbf{n}}^T \mathbf{I}\, \hat{\mathbf{n}}$$

as the moment of inertia about the axis $\hat{\mathbf{n}}$. This corresponds to what we called I earlier, when we didn't make explicit reference to the rotation axis we were using. Here we are thinking of a matrix notation, so $\hat{\mathbf{n}}^T$ means the transpose of $\hat{\mathbf{n}}$, which gives a row vector.

The result $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ shows quite clearly that although the angular momentum depends linearly on $\boldsymbol{\omega}$ it does *not* have to be parallel to $\boldsymbol{\omega}$. One important place where this matters is wheel balancing on cars. A wheel is unbalanced precisely when $\mathbf{L}$ and $\boldsymbol{\omega}$ are not parallel. Then, as the wheel rotates with $\boldsymbol{\omega}$ fixed, $\mathbf{L}$ describes a cone so $d\mathbf{L}/dt \neq 0$. Therefore a torque must be applied and you feel "wheel wobble." This is corrected by adding small masses to the wheel rim to adjust $\mathbf{I}$ to make $\mathbf{L}$ and $\boldsymbol{\omega}$ line up. In general, since $\mathbf{I}$ is a symmetric matrix, it can be diagonalised. This means it is always possible to choose a set of axes in the body for which $\mathbf{I}$ has non zero elements only along the diagonal. If you rotate the body around one of these *principal axes*, $\mathbf{L}$ and $\boldsymbol{\omega}$ will be parallel.

2.7.1 Free Rotation of a Rigid Body — Geometric Description*

Consider the rotational motion of a rigid body moving freely under no forces (or, a rigid body falling freely in a uniform gravitational field so that there are no torques about the CM; or, a rigid body freely pivoted at the CM).

If there are no torques acting, the total angular momentum, $\mathbf{L}$, must remain constant. It is convenient to choose axes fixed in the body, aligned with its principal axes of inertia. These body axes are themselves rotating, so in these coordinates the components of $\mathbf{L}$ along the axes may change (see chapter 4 on rotating coordinate systems). However, $|\mathbf{L}|$ is still fixed, so that $\mathbf{L}\cdot\mathbf{L} = L^2 = \text{const}$. Expressed in the body coordinates, this reads:

$$L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2.$$

Furthermore, since there is no torque, the rotational kinetic energy is fixed, $T = \text{const}$. Expressed in the body coordinates, this second conservation condition reads:

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2.$$

The components of the angular velocity simultaneously satisfy two different equations. These equations specify two ellipsoids and $\boldsymbol{\omega}$ must lie on the line given by their intersection.

Suppose that all three principal moments of inertia are unequal, as is the case for, say, a book or a tennis racket. We'll take $I_1 < I_2 < I_3$. Now, start spinning the object with angular velocity of magnitude ω aligned along the I_1 axis. Angular momentum conservation says that the maximum magnitude of the component of $\boldsymbol{\omega}$ along the I_2 axis in the subsequent motion is $\omega I_1/I_2$, while kinetic energy conservation says the maximum magnitude of this component is $\omega\sqrt{I_1/I_2}$. Since $I_1 < I_2$, we find that the maximum component allowed by kinetic energy conservation is bigger, so that the kinetic energy ellipsoid lies *outside* the angular momentum ellipsoid along the I_2 axis. Likewise, since $I_1 < I_3$, the kinetic energy ellipsoid lies *outside* the angular momentum ellipsoid in the I_3 direction. Therefore, the intersection of the two ellipsoids comprises just two points, along the positive and negative I_1 directions. This is enough to tell you that rotation about the I_1 axis is stable — see figure 2.9(a).

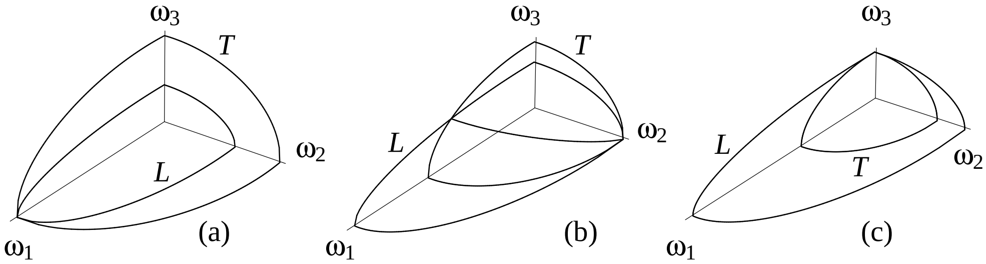


Figure 2.9 Free rotation of a rigid body. The diagrams show the (first octants of the) kinetic energy and angular momentum ellipsoids for the free rotation of a rigid body with all three principal moments of inertia different, $I_1 < I_2 < I_3$. In (a) the rotation is stable with $\boldsymbol{\omega}$ pointing along the I_1 direction. In (b) the two ellipsoids intersect in a line, showing that rotation about the I_2 axis is unstable. In (c) the rotation is stable with $\boldsymbol{\omega}$ pointing in the I_3 direction.

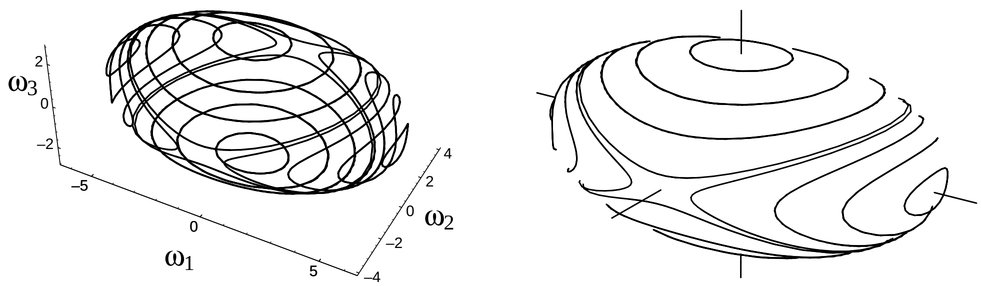


Figure 2.10 Curves showing the time variation of angular velocity for a freely rotating object. The curves all lie on the ellipsoid of constant kinetic energy, and each one is given by the intersection of this ellipsoid with a similar ellipsoid of constant (magnitude of) angular momentum. On the left the full curves are shown, while on the right, parts of the curves on the "back" of the kinetic energy ellipsoid are hidden. The closed loops around the I_1 and I_3 axes show that the rotation is stable about these two axes.

A similar argument holds if you start with the angular velocity lined up along the I_3 axis, although in this case the angular momentum ellipsoid lies outside the kinetic energy ellipsoid, with the intersection only at two points along the positive and negative I_3 axes. Thus, rotation about the axis with the largest moment of inertia is also stable — see figure 2.9(c).

The final case we consider is where the initial angular velocity is aligned along the I_2 axis. Now, since $I_2 > I_1$, the angular momentum ellipsoid lies outside the kinetic energy ellipsoid in the I_1 direction, but, since $I_2 < I_3$, the angular momentum ellipsoid lies inside the kinetic energy ellipsoid in the I_3 direction. This means that there is a whole line of points where the two ellipsoids intersect — see figure 2.9(b). In turn, this tells you that rotation about the axis with intermediate moment of inertia is unstable: any small misalignment can be amplified and the object will be observed to "tumble" as it spins. It is easy to demonstrate this for yourself by throwing a book in the air, spinning it about each of its three principal axes in turn.

These three cases are illustrated in figure 2.9. Figure 2.10 shows the time variation of ω for the freely rotating body: each continuous curve shows the time variation of the components of ω. The curves all lie on the surface of the ellipsoid of constant kinetic energy, and each curve is given by the intersection of this ellipsoid with an ellipsoid of constant angular momentum.

3

Gravitation and Kepler's Laws

In this chapter we will recall the law of universal gravitation and will then derive the result that a spherically symmetric object acts gravitationally like a point mass at its centre if you are outside the object. Following this we will look at orbits under gravity, deriving Kepler's laws. The chapter ends with a consideration of the energy in orbital motion and the idea of an effective potential.

3.1 Newton's Law of Universal Gravitation

For two particles of masses m_1 and m_2 separated by distance r there is a mutual force of attraction of magnitude

$$\frac{Gm_1 m_2}{r^2},$$

where $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is the *gravitational constant*. If $\mathbf{F}_{12}$ is the force of particle 2 on particle 1 and vice-versa, and if $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ is the vector from particle 1 to particle 2, as shown in figure 3.1, then the vector form of the law is:

$$\boxed{\mathbf{F}_{12} = -\mathbf{F}_{21} = \frac{Gm_1 m_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12}},$$

where the hat ($\hat{}$) denotes a unit vector as usual. Gravity obeys the superposition principle, so if particle 1 is attracted by particles 2 and 3, the total force on 1 is $\mathbf{F}_{12} + \mathbf{F}_{13}$.

The gravitational force is exactly analogous to the electrostatic Coulomb force if you make the replacements, $m \to q$, $-G \to 1/4\pi\varepsilon_0$ (of course, masses are always

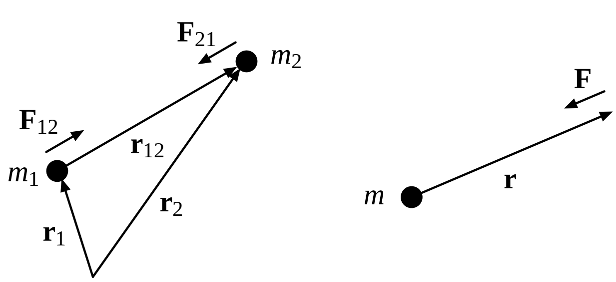


Figure 3.1 Labelling for gravitational force between two masses (left) and gravitational potential and field for a single mass (right).

positive, whereas charges q can be of either sign). We will return to this analogy later.

Since gravity acts along the line joining the two masses, it is a *central force* and therefore *conservative* (any central force is conservative — why ?). For a conservative force, you can sensibly define a *potential energy difference* between any two points according to,

$$V(\mathbf{r}_f) - V(\mathbf{r}_i) = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} d\cdot\mathbf{r}.$$

The definition is sensible because the answer depends only on the endpoints and not on which particular path you used. Since only *differences* in potential energy appear, we can arbitrarily choose a particular point, say $\mathbf{r}_0$, as a reference and declare its potential energy to be zero, $V(\mathbf{r}_0) = 0$. If you're considering a planet orbiting the Sun, it is conventional to set $V = 0$ at infinite separation from the Sun, so $|\mathbf{r}_0| = \infty$. This means that we can define a gravitational potential energy by making the conventional choice that the potential is zero when the two masses are infinitely far apart. For convenience, let's put the origin of coordinates at particle 1 and let $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ be the position of particle 2. Then the gravitational force on particle 2 due to particle 1 is $\mathbf{F} = \mathbf{F}_{21} = -Gm_1m_2\,\hat{\mathbf{r}}/r^2$ and the gravitational potential energy is,

$$\boxed{V(r) = -\int_\infty^r \mathbf{F}\cdot d\mathbf{r}' = -\int_\infty^r (-\frac{Gm_1m_2}{r'^2})dr' = -\frac{Gm_1m_2}{r}}.$$

(The prime ($'$) on the integration variable is simply to distinguish it from the point where we are evaluating the potential energy.) It is also useful to think of particle 1 setting up a gravitational field which acts on particle 2, with particle 2 acting as a test mass for probing the field. Define the *gravitational potential*, which is the gravitational potential energy per unit mass, for particle 1 by (setting $m_1 = m$ now),

$$\boxed{\Phi(r) = -\frac{Gm}{r}}.$$

Likewise, define the *gravitational field* $\mathbf{g}$ of particle 1 as the gravitational force per unit mass:

$$\boxed{\mathbf{g}(\mathbf{r}) = -\frac{Gm}{r^2}\,\hat{\mathbf{r}}}.$$

The use of $\mathbf{g}$ for this field is deliberate: the familiar $g = 9.81\,\mathrm{m\,s^{-2}}$ is just the magnitude of the Earth's gravitational field at its surface. The field and potential are related in the usual way:

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi.$$

Gravitational Potential Energy Near the Earths' Surface If you are thinking about a particle moving under gravity near the Earth's surface, you might set the $V = 0$ at the surface. Here, the gravitational force on a particle of mass m is,

$$\mathbf{F} = -mg\,\hat{\mathbf{k}},$$

where $\hat{\mathbf{k}}$ is an upward vertical unit vector, and $g = 9.81\,\mathrm{m\,s^{-2}}$ is the magnitude of the gravitational acceleration. In components, $F_x = F_y = 0$ and $F_z = -mg$. Since the force is purely vertical, the potential energy is independent of x and y. We will measure z as the height above the surface. Applying the definition of potential energy difference between height h and the Earth's surface ($z=0$), we find

$$V(h)-V(0)=-\int_0^h F_z\,dz=-\int_0^h(-mg)\,dz=mgh.$$

Choosing $z=0$ as our reference height, we set $V(z{=}0)=0$ and find the familiar result for gravitational potential energy,

$$\boxed{V(h)=mgh \qquad \text{Gravitational potential energy near the Earth's surface}}$$

Note that since the gravitational force acts vertically, on any path between two given points the work done by gravity depends only on the changes in height between the endpoints. So, this force is indeed conservative.

3.2 Gravitational Attraction of a Spherical Shell

The problem of determining the gravitational attraction of spherically symmetric objects led Newton to invent calculus: it took him many years to prove the result. The answer for a thin uniform spherical shell of matter is that outside the shell the gravitational force is the same as that of a point mass of the same total mass as the shell, located at the centre of the shell. Inside the shell, the force is zero. By considering an arbitrary spherically symmetric object to be built up from thin shells, we immediately find that outside the object the gravitational force is the same as that of a point with the same total mass located at the centre.

We will demonstrate this result in two ways: first by calculating the gravitational potential directly, and then, making full use of the spherical symmetry, using the analogy to electrostatics and applying Gauss' law.

3.2.1 Direct Calculation

We consider a thin spherical shell of radius a, mass per unit area ρ and total mass $m=4\pi\rho a^2$. Use coordinates with origin at the center of the shell and calculate the gravitational potential at a point P distance r from the centre as shown in figure 3.2.

We use the superposition principle to sum up the individual contributions to the potential from all the mass elements in the shell. All the mass in the thin annulus of width $a\,d\theta$ at angle θ is at the same distance R from P, so we can use this as our element of mass:

$$dm=\rho\,2\pi a\sin\theta\,a d\theta=\frac{m}{2}\sin\theta\,d\theta.$$

The contribution to the potential from the annulus is,

$$d\Phi=-\frac{G\,dm}{R}=-\frac{Gm}{2}\frac{\sin\theta\,d\theta}{R}.$$

Now we want to sum all the contributions by integrating over θ from 0 to π. In fact, it is convenient to change the integration variable from θ to R. They are related using the cosine rule:

$$R^2=r^2+a^2-2ar\cos\theta.$$

From this we find $\sin\theta\,d\theta/R=dR/(ar)$, which makes the integration simple. If $r\geq a$ the integration limits are $r-a$ and $r+a$, while if $r\leq a$ they are $a-r$ and $a+r$.

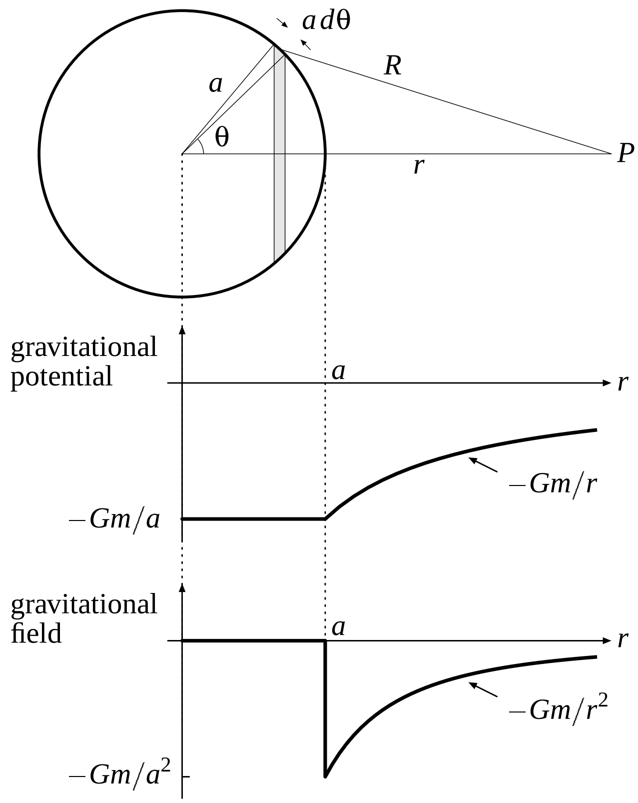


Figure 3.2 Gravitational potential and field for a thin uniform spherical shell of matter.

We can specify the limits for both cases as $|r-a|$ and $r+a$, so that:

$$\Phi(r) = -\frac{Gm}{2ar}\int_{|r-a|}^{r+a} dR = \begin{cases} -Gm/r & \text{for } r \geq a \\ -Gm/a & \text{for } r < a \end{cases}.$$

We obtain the gravitational field by differentiating:

$$\mathbf{g}(\mathbf{r}) = \begin{cases} -Gm\hat{\mathbf{r}}/r^2 & \text{for } r \geq a \\ 0 & \text{for } r < a \end{cases}.$$

As promised, outside the shell, the potential is just that of a point mass at the centre. Inside, the potential is constant and so the force vanishes. The immediate corollaries are:

- A uniform or spherically stratified sphere (so the density is a function of the radial coordinate only) attracts like a point mass of the same total mass at its centre, when you are outside the sphere;
- Two non-intersecting spherically symmetric objects attract each other like two point masses at their centres.

3.2.2 The Easy Way

Now we make use of the equivalence of the gravitational force to the Coulomb force using the relabelling summarised in table 3.1. We can now apply the integral form of Gauss' Law in the gravitational case to our spherical shell. The law reads,

$$\int_S \mathbf{g}\cdot d\mathbf{S} = -4\pi G \int_V \rho_m\, dV$$

Coulomb force		Gravitational force	
charge	q	mass	m
coupling	$1/(4\pi\varepsilon_0)$	coupling	$-G$
potential	V	potential	Φ
electric field	$\mathbf{E} = -\boldsymbol{\nabla} V$	gravitational field	$\mathbf{g} = -\boldsymbol{\nabla}\Phi$
charge density	ρ_q	mass density	ρ_m
Gauss' law	$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho_q/\varepsilon_0$	Gauss' law	$\boldsymbol{\nabla}\cdot\mathbf{g} = -4\pi G\rho_m$

Table 3.1 Equivalence between electrostatic Coulomb force and gravitational force.

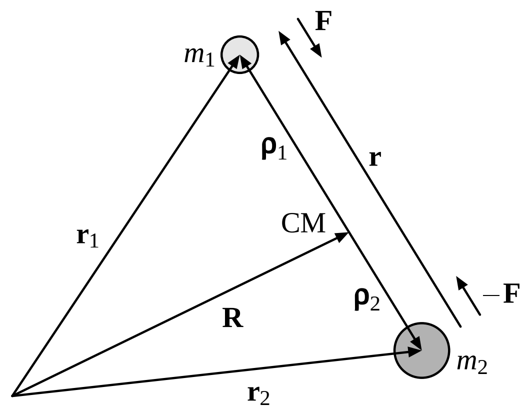


Figure 3.3 Coordinates for a two-body system.

which says that the surface integral of the normal component of the gravitational field over a given surface S is equal to $(-4\pi G)$ times the mass contained within that surface, with the mass obtained by integrating the mass density ρ_m over the volume V contained by S.

The spherical symmetry tells us that the gravitational field $\mathbf{g}$ must be radial, $\mathbf{g} = g\,\hat{\mathbf{r}}$. If we choose a concentric spherical surface with radius $r > a$, the mass enclosed is just m, the mass of the shell, and Gauss' Law says,

$$4\pi r^2 g = -4\pi G m$$

which gives

$$\mathbf{g} = -\frac{Gm}{r^2}\,\hat{\mathbf{r}} \qquad \text{for } r > a$$

immediately. Likewise, if we choose a concentric spherical surface inside the shell, the mass enclosed is zero and $\mathbf{g}$ must vanish.

3.3 Orbits: Preliminaries

3.3.1 Two-body Problem: Reduced Mass

Consider a system of two particles of masses m_1 at position $\mathbf{r}_1$ and m_2 at $\mathbf{r}_2$ interacting with each other by a conservative central force, as shown in figure 3.3. We imagine these two particle to be isolated from all other influences so that there is no external force.

Express the position $\mathbf{r}_i$ of each particle as the centre of mass location $\mathbf{R}$ plus a displacement $\boldsymbol{\rho}_i$ relative to the centre of mass, as we did in equation (1.3) in chapter 1 on page 2.

$$\mathbf{r}_1 = \mathbf{R} + \boldsymbol{\rho}_1, \qquad \mathbf{r}_2 = \mathbf{R} + \boldsymbol{\rho}_2.$$

Now change variables from $\mathbf{r}_1$ and $\mathbf{r}_2$ to $\mathbf{R}$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Since the only force acting is the internal force, $\mathbf{F} = \mathbf{F}_{12} = -\mathbf{F}_{21}$, between particles 1 and 2, the equations of motion are:

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}, \qquad m_2\ddot{\mathbf{r}}_2 = -\mathbf{F}.$$

From these we find, setting $M = m_1 + m_2$,

$$M\ddot{\mathbf{R}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0,$$

which says that the centre of mass moves with constant velocity, as we already know from the general analysis in section 1.1.1 (see page 2). For the new relative displacement $\mathbf{r}$, we find,

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F} = \frac{m_1+m_2}{m_1 m_2}\mathbf{F},$$

which we write as,

$$\boxed{\mathbf{F} = \mu\ddot{\mathbf{r}}}, \tag{3.1}$$

where we have defined the *reduced mass*

$$\boxed{\mu \equiv \frac{m_1 m_2}{m_1+m_2}}.$$

For a conservative force $\mathbf{F}$ there is an associated potential energy $V(r)$ and the total energy of the system becomes

$$E = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 + V(r).$$

This is just an application of the general result we derived for the kinetic energy of a system of particles in equation (1.4) on page 3 — we already applied it in the two-particle case on page 3. Likewise, when $\mathbf{F}$ is central, the angular momentum of the system is

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \mu\mathbf{r} \times \dot{\mathbf{r}},$$

which is an application of the result in equation (1.6) on page 7. You should make sure you can reproduce these two results.

Since the center of mass $\mathbf{R}$ moves with constant velocity we can switch to an inertial frame with origin at $\mathbf{R}$, so that $\mathbf{R} = 0$. Then we have:

$$\boxed{\begin{aligned} E &= \frac{1}{2}\mu\dot{\mathbf{r}}^2 + V(r), \\ \mathbf{L} &= \mu\mathbf{r} \times \dot{\mathbf{r}}. \end{aligned}} \tag{3.2}$$

The original two-body problem reduces to an equivalent problem of a single body of mass μ at position vector $\mathbf{r}$ relative to a fixed centre, acted on by the force $\mathbf{F} = -(\partial V/\partial r)\,\hat{\mathbf{r}}$.

It's often the case that one of the masses is very much larger than the other, for example:

$$\begin{aligned} m_{\text{Sun}} &\gg m_{\text{planet}}, \\ m_{\text{Earth}} &\gg m_{\text{satellite}}, \\ m_{\text{proton}} &\gg m_{\text{electron}}. \end{aligned}$$

If $m_2 \gg m_1$, then $\mu = m_1 m_2/(m_1+m_2) \approx m_1$ and the reduced mass is nearly equal to the light mass. Furthermore,

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2} \approx \mathbf{r}_2$$

and the centre of mass is effectively at the larger mass. In such cases we treat the larger mass as fixed at $\mathbf{r}_2 \approx 0$, with the smaller mass orbiting around it, and set μ equal to the smaller mass. This is sometimes called the "fixed Sun and moving planet approximation." We will use this approximation when we derive Kepler's Laws. We will also ignore interactions between planets in comparison to the gravitational attraction of each planet towards the Sun.

3.3.2 Two-body Problem: Conserved Quantities

Recall that gravity is a central force: the gravitational attraction between two bodies acts along the line joining them. In the formulation of equations 3.2 above, this means that the gravitational force on the mass μ acts in the direction $-\mathbf{r}$ and therefore exerts no torque about the fixed centre. Consequently, the angular momentum vector $\mathbf{L}$ is a constant: its magnitude is fixed and it points in a fixed direction. Since $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ (where $\mathbf{p} = \mu\dot{\mathbf{r}}$), we see that $\mathbf{L}$ is always perpendicular to the plane defined by the position and momentum of the mass μ. Alternatively stated, this means that $\mathbf{r}$ and $\mathbf{p}$ must always lie in the fixed plane of all directions perpendicular to $\mathbf{L}$, and can therefore be described using plane polar coordinates (r,θ), with origin at the fixed centre.

For completeness we quote the radial and angular equations of motion in these plane polar coordinates. We set the reduced mass equal to the planet's mass m and write the gravitational force as $\mathbf{F} = -k\hat{\mathbf{r}}/r^2$, where $k = GMm$ and M is the Sun's mass. The equations become (the reader should exercise to reproduce the following expressions):

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{k}{mr^2} && \text{radial equation,} \\ \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) &= 0 && \text{angular equation.} \end{aligned}$$

The angular equation simply expresses the conservation of the angular momentum $L = mr^2\dot{\theta}$.

The second conserved quantity is the total energy, kinetic plus potential. All central forces are conservative and in our two-body orbit problem the only force acting is the central gravitational force. We again set μ equal to the planet's mass m and write the gravitational potential energy as $V(r) = -k/r$. Then the expression for the constant total energy becomes, using plane polar coordinates,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - k/r.$$

In section 3.5 on page 33 we will deduce a good deal of information about the orbit straight from this conserved total energy.

3.3.3 Two-body Problem: Examples

Comet A comet approaching the Sun in the plane of the Earth's orbit (assumed circular) crosses the orbit at an angle of 60° travelling at $50\,\mathrm{km\,s^{-1}}$. Its closest approach to the Sun is $1/10$ of the Earth's orbital radius. Calculate the comet's speed at the point of closest approach.

Take a circular orbit of radius r_e for the Earth. Ignore the attraction of the comet to the Earth compared to the attraction of the comet to the Sun and ignore any complications due to the reduced mass.

The key to this problem is that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$ of the comet about the Sun is fixed. At the point of closest approach the comet's velocity must be tangential only (why?), so that,

$$|\mathbf{r} \times \mathbf{v}| = r_{\mathrm{min}} v_{\mathrm{max}}.$$

At the crossing point,

$$|\mathbf{r} \times \mathbf{v}| = r_e v \sin 30°.$$

Equating these two expressions gives,

$$r_{\mathrm{min}} v_{\mathrm{max}} = 0.1\, r_e v_{\mathrm{max}} = \frac{1}{2} r_e v,$$

leading to

$$v_{\mathrm{max}} = 5v = 250\,\mathrm{km\,s^{-1}}.$$

Cygnus X1 Cygnus X1 is a binary system of a supergiant star of 25 solar masses and a black hole of 10 solar masses, each in a circular orbit about their centre of mass with period 5.6 days. Determine the distance between the supergiant and the black hole, given that a solar mass is 1.99×10^{30} kg.

Here we apply the two-body equation of motion, equation (3.1) from page 26. Labelling the two masses m_1 and m_2, their separation r and their angular velocity ω, we have,

$$\frac{G m_1 m_2}{r^2} = \frac{m_1 m_2}{m_1 + m_2} r \omega^2.$$

Rearranging and using the period $T = 2\pi/\omega$, gives

$$\begin{aligned} r^3 &= \frac{G(m_1 + m_2)T^2}{4\pi^2} \\ &= \frac{6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}} \times (10 + 25) \times 1.99 \times 10^{30}\,\mathrm{kg} \times (5.6 \times 86400\,\mathrm{s})^2}{4\pi^2} \\ &= 27.5 \times 10^{30}\,\mathrm{m}^3, \end{aligned}$$

leading to $r = 3 \times 10^{10}$ m.

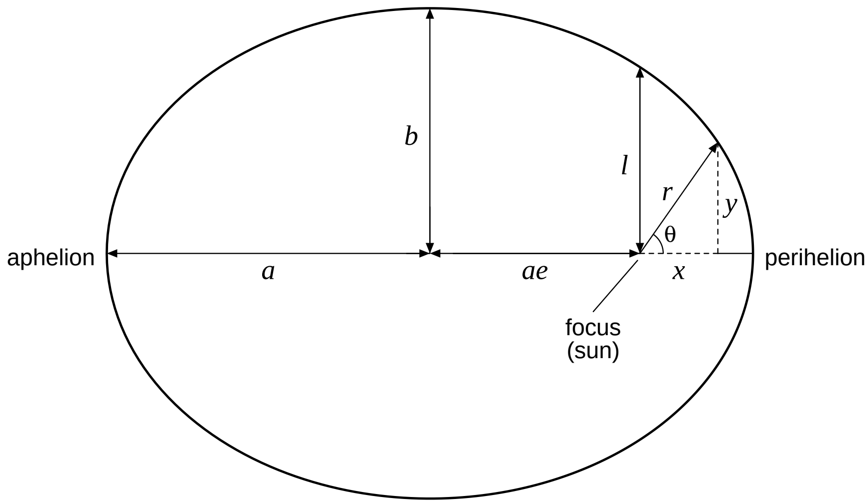
3.4 Kepler's Laws

3.4.1 Statement of Kepler's Laws

1. The orbits of the planets are ellipses with the Sun at one focus.

2. The radius vector from the Sun to a planet sweeps out equal areas in equal times.

3. The square of the orbital period of a planet is proportional to the cube of the semimajor axis of the planet's orbit ($T^2 \propto a^3$).



polar equation	$\frac{l}{r} = 1 + e\cos\theta$
cartesian equation	$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$
semimajor axis	$a = \frac{l}{1-e^2} = -\frac{k}{2E}$
semiminor axis	$b = \frac{l}{\sqrt{1-e^2}} = a^{1/2}l^{1/2}$
eccentricity	e
semi latus rectum	$l = \frac{L^2}{mk}$
constant k	$k = GMm$
total energy	$E = -\frac{mk^2}{2L^2}(1-e^2) = -\frac{k}{2a}$

Figure 3.4 Geometry of an ellipse and relations between its parameters. In the polar and cartesian equations for the ellipse, the origin of coordinates is at the *focus*.

3.4.2 Summary of Derivation of Kepler's Laws

We will be referring to the properties of ellipses, so figure 3.4 shows an ellipse and its geometric parameters. The parameters are also expressed in terms of the dynamical quantities: energy E, angular momentum L, mass of the Sun M, mass of the planet m and the universal constant of gravitation G. The semimajor axis a is fixed by the total energy E and the semi latus rectum l is fixed by the total angular momentum L.

In general the path of an object orbiting under an inverse square law force can be any conic section. This means that the orbit may be an ellipse with $0 \leq e < 1$, parabola with $e = 1$ or hyperbola with $e > 1$. With the definition that the zero of potential energy occurs for infinite separation, the total energy of the system is negative for an elliptical orbit. When the total energy is zero the object can just escape to infinite distance, where it will have zero kinetic energy: this is a parabolic orbit. For positive energy, the object can escape to infinite separation with finite kinetic energy: this gives a hyperbolic orbit. Figure 3.5 illustrates the possible orbital

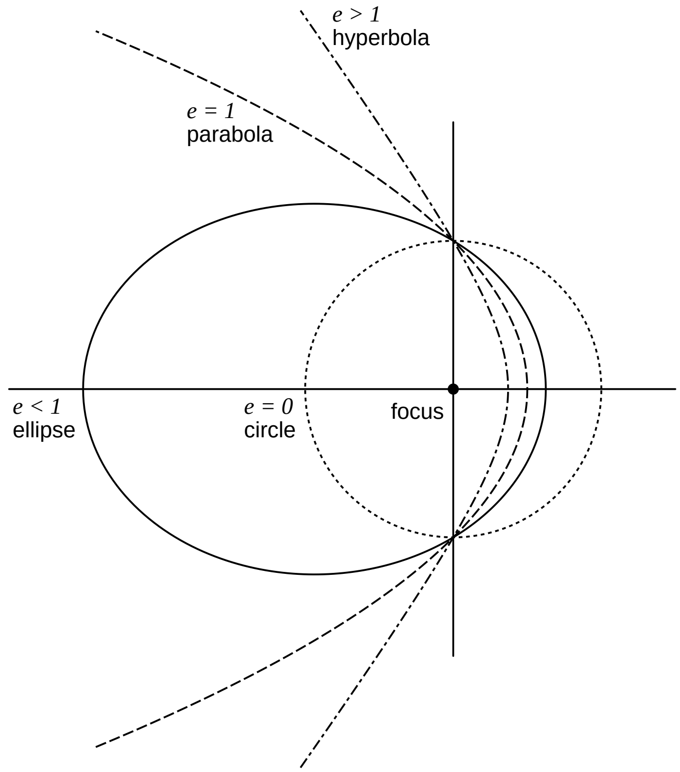


Figure 3.5 Different conic sections, showing possible orbits under an inverse square law force. The figure is drawn so that each orbit has the same angular momentum (same l) but different energy (the mass of the orbiting object is held fixed).

shapes.

2nd Law This is the most general and is a statement of angular momentum conservation under the action of the *central* gravitational force. The angular equation of motion gives:

$$r^2\dot{\theta} = \frac{L}{m} = \text{const.}$$

This immediately leads to,

$$\boxed{\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{const}}.$$

The 2nd law is illustrated in figure 3.6. An orbiting planet moves along the arc segments AB and CD in equal times, and the two shaded areas are equal.

Orbit equation The first and third laws are arrived at by finding the equation for the orbit. The fact that the orbits are ellipses is *specific* to an inverse square law for the force, and hence the first and third laws are also specific to an inverse square law force.

Proceed as follows, starting from the radial equation of motion (with $k = GMm$),

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{mr^2}.$$

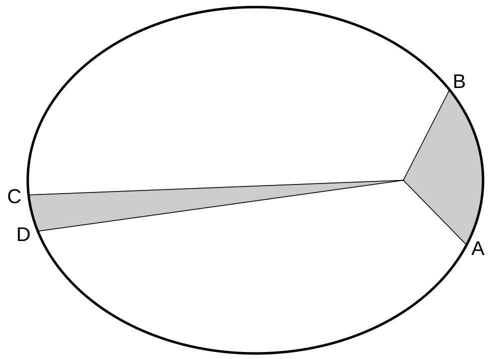


Figure 3.6 Illustration of Kepler's 2nd Law. An orbiting planet moves along the arc segments AB and CD in equal times, and the two shaded areas are equal.

(i) Eliminate $\dot{\theta}$ using angular momentum conservation, $\dot{\theta} = L/mr^2$, leading to a differential equation for r alone:

$$\ddot{r} - \frac{L^2}{m^2 r^3} = -\frac{k}{mr^2}.$$

(ii) Use the relation

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} = \frac{L}{mr^2}\frac{d}{d\theta},$$

to obtain derivatives with respect to θ in place of time derivatives. This gives a differential equation for r in terms of θ.

(iii) To obtain an equation which is easy to solve, make the substitution $u = 1/r$, to obtain the orbit equation:

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{mk}{L^2}}.$$

1st Law The solution of the orbit equation is

$$\frac{1}{r} = \frac{mk}{L^2}(1 + e\cos\theta),$$

which for $0 \le e < 1$ gives an ellipse, with semi latus rectum $l = L^2/mk$. This is the first law.

In figure 3.7 we show the orbit of a hypothetical planet around the Sun with semimajor axis 1.427×10^9 km (the same as Saturn) and eccentricity $e = 0.56$ (bigger than for any real planet — Pluto has the most eccentric orbit with $e = 0.25$). The figure also shows how the planet's distance from the Sun, speed and angular velocity vary during its orbit.

3rd Law Start with the 2nd law for the rate at which area is swept out,

$$\frac{dA}{dt} = \frac{L}{2m},$$

and integrate over a complete orbital period T, to give $T = 2mA/L$, where $A = \pi ab$ is the area of the ellipse. Substituting for b in terms of a gives the third law:

$$\boxed{T^2 = \frac{4\pi^2}{GM}a^3}.$$

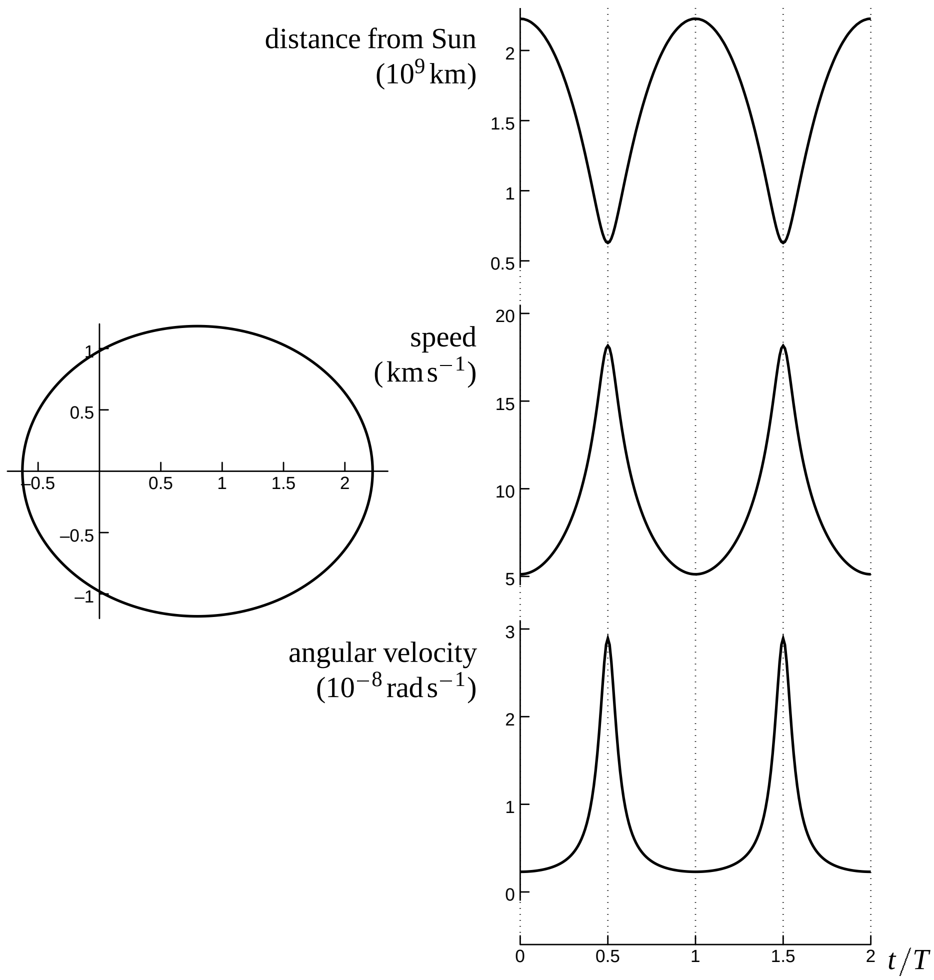


Figure 3.7 On the left is shown the orbit of a hypothetical planet around the Sun with distance scales marked in units of 10^9 km. The planet has the same semimajor axis $a = 1.427 \times 10^9$ km as Saturn, and hence the same period, $T = 10760$ days. The eccentricity is $e = 0.56$. The three graphs on the right show the planet's distance from the sun, speed and angular velocity respectively as functions of time measured in units of the orbital period T.

Kepler's Procedure* The solution of the orbit equation gives r as a function of θ, but if you're an astronomer, you may well be interested in knowing $\theta(t)$, so that you can track a planet's position in orbit as a function of time. You could do this by brute force by combining the angular equation of motion, $r^2\dot{\theta} = L/m$, with the equation giving the orbit, $l/r = 1 + e\cos\theta$, and integrating. This gives a disgusting integral which moreover leads to t as a function of θ: you have to invert this, by a series expansion method, to get θ as a function of t. This is tedious, and requires you to keep many terms in the expansion to match the accuracy of astronomical observations. Kepler himself devised an ingenious geometrical way to determine $\theta(t)$, and his construction leads to a much neater numerical procedure. I refer you to the textbook by Marion and Thornton[1] for a description.

[1] J B Marion and S T Thornton, *Classical Dynamics of Particles and Systems*, 3rd edition, Harcourt Brace Jovanovich (1988) p261

3.4.3 Scaling Argument for Kepler's 3rd Law

Suppose you have found a solution of the orbit equation, $\ddot{r} - r\dot{\theta}^2 = -k/mr^2$, giving r and θ as functions of t. Now scale the radial and time variables by constants α and β respectively:

$$r' = \alpha r, \qquad t' = \beta t.$$

In terms of the new variables r' and t', the left hand side of the orbit equation becomes,

$$\frac{d^2r'}{dt'^2} - r'\left(\frac{d\theta}{dt'}\right)^2 = \frac{\alpha}{\beta^2}\ddot{r} - \alpha r\left(\frac{\dot{\theta}}{\beta}\right)^2 = \frac{\alpha}{\beta^2}(\ddot{r} - r\dot{\theta}^2),$$

while the right hand side becomes,

$$-\frac{k}{mr'^2} = \frac{1}{\alpha^2}\left(-\frac{k}{mr^2}\right).$$

Comparing the two sides, you can see that we will have a new solution in terms of r' and t' provided $\beta^2 = \alpha^3$. But this says precisely that if you have orbits of similar shape, the period T and semimajor axis a (characterising the linear size of the orbit) will be related by $T^2 \propto a^3$, which is Kepler's third law.

To find the constant of proportionality and show that the orbits are conic sections, you really have to solve the orbit equation. However, the scaling argument makes clear how the third law depends on having an inverse-square force law.

3.5 Energy Considerations: Effective Potential

Since the gravitational force is conservative, the total energy E of the orbiting body is conserved. Writing $V(r)$ for the gravitational potential energy for a moment (so that we can substitute different forms for the potential energy if necessary), we find

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r).$$

Since we know that angular momentum is also conserved (the force is central), we can eliminate $\dot{\theta}$ using $r^2\dot{\theta} = L/m$, to leave,

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r)}.$$

This is just the energy equation you would get for a particle moving in one dimension in an *effective potential*

$$\boxed{U(r) = \frac{L^2}{2mr^2} + V(r)}.$$

The effective potential contains an additional *centrifugal term*, $L^2/2mr^2$, which arises because angular momentum has to be conserved. We can learn a good deal about the possible motion by studying the effective potential without having to solve the equation of motion for r.

In our case, replacing $V(r)$ by the gravitational potential energy and using $l = L^2/mk$, the effective potential becomes (see figure 3.8)

$$U(r) = \frac{kl}{2r^2} - \frac{k}{r}.$$

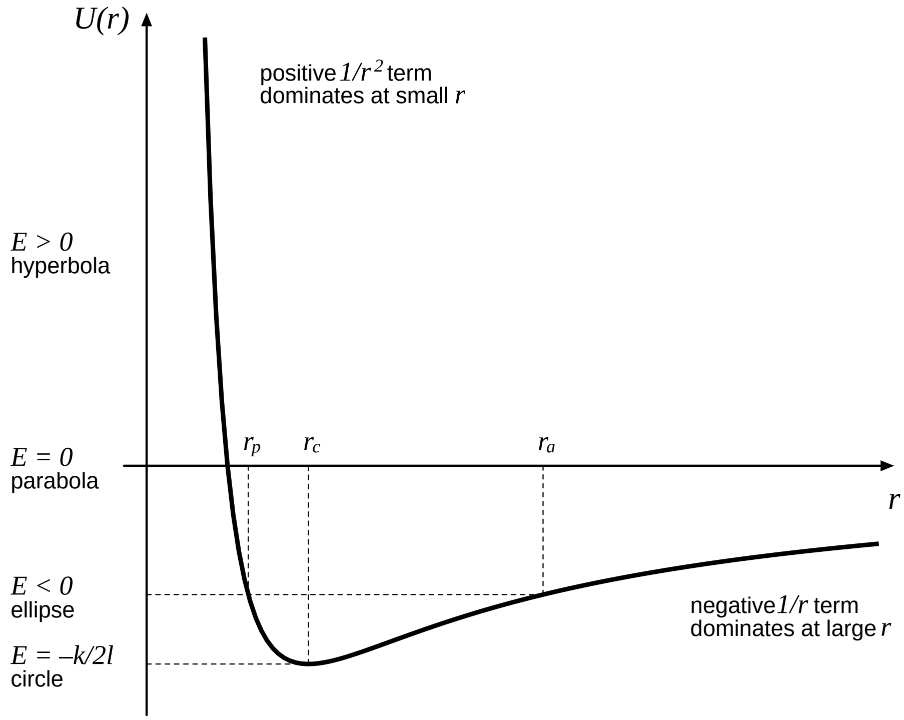


Figure 3.8 Effective potential $U(r) = kl/2r^2 - k/r$ for motion in an inverse-square law force.

The allowed motion must have $\dot{r}^2 \geq 0$, so the energy equation says

$$E \geq U(r) = \frac{kl}{2r^2} - \frac{k}{r}.$$

If we choose a value for the total energy E, we can then draw a horizontal line at this value on the graph of $U(r)$, and we know that the allowed motion occurs only where the $U(r)$ curve lies *below* our chosen value of E.

The minimum possible total energy (for a given angular momentum) is given by the minimum of the curve of $U(r)$. In this situation r is constant at

$$r_c = l = L^2/mk,$$

so the orbit is a circle and the total energy is $E = -k/2l = -mk^2/2L^2$.

If $-k/2l < E < 0$, you can see that the motion is allowed for a finite range of r, $r_p \leq r \leq r_a$. This is the case of an elliptical orbit with perihelion r_p and aphelion r_a. You can find the values of r_p and r_a by finding the roots of the equation $E = kl/2r^2 - k/r$.

If $E = 0$, you see that there is a minimum value for r, but that escape to infinity is just possible. This is the case of a parabolic orbit. For $E > 0$, escape to infinity is possible with finite kinetic energy at infinite separation. This is the case of a hyperbolic orbit.

Orbits in a Yukawa Potential We found that the orbits produced by an inverse-square law attractive force were ellipses, where the planet repeatedly traced the same path through space. Now consider a force given by the Yukawa potential,

$$V(r) = -\frac{\alpha e^{-\kappa r}}{r} \qquad (\alpha > 0, \kappa > 0).$$

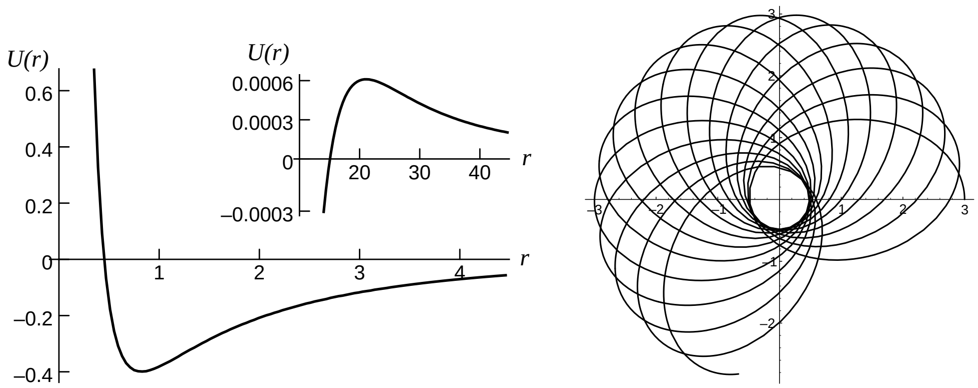


Figure 3.9 Left: effective potential $U(r) = L^2/2mr^2 - \alpha e^{-\kappa r}/r$ with $m = 1$, $\alpha = 1$, $\kappa = 0.24$ and $L = 0.9$. The inset shows $U(r)$ at large r where it has a local maximum (note the differences in scale, particularly for the value of U). Right: rosette orbit of a particle with this effective potential.

Such a potential describes, for example, the force of attraction between nucleons in an atomic nucleus. Of course, in that situation, the problem should be treated quantum mechanically, but for now, let's just look at classical orbits under the influence of this potential.

The effective potential is,

$$U(r) = \frac{L^2}{2mr^2} - \frac{\alpha e^{-\kappa r}}{r}.$$

To be specific, work in dimensionless units, setting $m = 1$, $\alpha = 1$, $\kappa = 0.24$ and choosing $L = 0.9$. The shape of the resulting effective potential as a function of r is shown in the left hand part of figure 3.9.

If the total energy E is negative but greater than the minimum of $U(r)$, then motion is allowed between a minimum and maximum value of the radius r. On the right hand side of figure 3.9 is the trajectory of a particle starting at $(x, y) = (3, 0)$ with $(v_x, v_y) = (0, 0.3)$ (so that $L = 0.9$). Here the particle's (dimensionless) energy is -0.117 and the motion is restricted to the region $0.486 \leq r \leq 3$, where 0.486 and 3 are the two solutions of the equation $U(r) = -0.117$.

Note that if $\kappa = 0$, the Yukawa potential reduces to the same form as the standard gravitational potential. So, if κr remains small compared to 1 we expect the situation to be a small perturbation relative to the gravitational case. In our example, for the "rosette" orbit on the right of figure 3.9, this is the case, and you can see that the orbit looks like an ellipse whose orientation slowly changes. This is often denoted "precession of the perihelion" and is typical of the effect of small perturbations on planetary orbits, for example those due to the effects of other planets. In fact, observed irregularities in the motion of Uranus led to the discovery of Neptune in 1846. The orientation of the major axis of the Earth's orbit drifts by about 104 seconds of arc each century, mostly due the influence of Jupiter. For Mercury, the perihelion advances by about 574 seconds of arc per century: 531 seconds of this can be explained by the Newtonian gravitational interactions of the other planets, while the remaining 43 seconds of arc are famously explained by Einstein's general relativity.

The effective potential shown in figure 3.9 displays another interesting property. At large r the $L^2/2mr^2$ dominates the exponentially falling Yukawa term, so $U(r)$ becomes positive. In our example, $U(r)$ has a local maximum near $r = 20$. If the

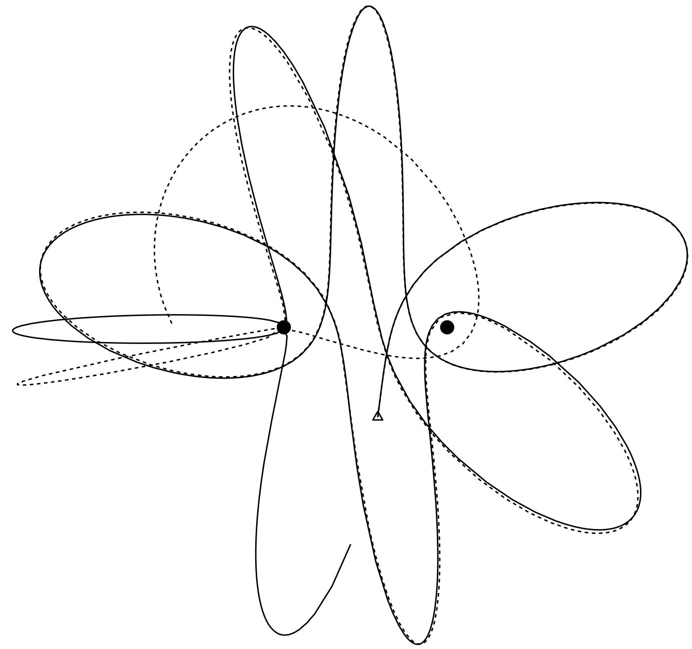


Figure 3.10 Orbital trajectories for a planet around two equal mass stars.

total energy is positive, but less than the value of U at the local maximum, there are two possibilities for orbital motion. For example, if $E = 0.0003$, we find either $0.451 \leq r \leq 16.31$ or $r \geq 36.48$. Classically these orbits are distinct, and a particle with $E = 0.0003$ which starts out in the inner region can never surmount the "barrier" in $U(r)$ and so will never be found in $r \geq 36.48$. In quantum mechanics, however, it is possible for a particle to "tunnel" through such a barrier, so that an initially bound particle has a (small) finite probability of escaping to large r. This is the case for a process like alpha decay.

3.6 Chaos in Planetary Orbits*

We have shown that a single planet orbiting the Sun follows a simple closed elliptical path. You might think that adding one more object to the system would make the equations more complicated, but that with patience and effort you might be able to figure out a solution for the trajectories. In fact, such a "three body problem" is notoriously intractable, and, even today, analytic solutions are known only in a few special cases.

In figure 3.10 is shown a numerical solution for a restricted version of the three body problem. The two black dots are stars of equal mass, held at fixed positions. This means that the total energy is conserved, but that the linear and angular momentum are not conserved since forces and torques have to be applied to hold the stars in place. The solid curve shows the trajectory of a planet which starts out with some given initial velocity at the point marked by the triangle. The stars are taken to have a finite radius and the planet is allowed to pass through them without suffering any interaction apart from the gravitational force (this avoids some numerical instability when the planet gets very close to a point mass). The complexity of the solid curve already hints at the difficulty of this problem.

In fact, the motion is chaotic in the scientific sense. One aspect of this is shown by the dashed curve. This is a second solution for a planet which also starts out at the point marked by the triangle, but has one of its initial velocity components differing by 0.5% from the corresponding component for the first case. You can see how the paths stay close together for a little while, but then rapidly diverge and show qualitatively different behaviour. This extreme (exponential) sensitivity to the initial conditions is one of the characteristics of chaotic systems. Contrast it to the two body problem, where a small perturbation to an elliptical orbit would simply result in a new slightly displaced orbit.

For an animated computer simulation of the three body problem described here, together with many other instructive examples of chaotic systems, try the program *Chaos Demonstrations* by J C Sprott and G Rowlands, available from Physics Academic Software, http://www.aip.org/pas/.

4

Rotating Coordinate Systems

4.1 Time Derivatives in a Rotating Frame

First recall the result that, for a vector $\mathbf{A}$ of *fixed length*, rotating about the origin with constant angular velocity $\boldsymbol{\omega}$, the rate of change of $\mathbf{A}$ is

$$\boxed{\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega}\times\mathbf{A}}.$$

Now let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ be unit vectors of an inertial frame O and let $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ be unit vectors of a rotating frame O'. Each of the primed basis vectors rotates rigidly with O', so

$$\frac{d\hat{\mathbf{i}}'}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{i}}',$$

with similar equations for $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$. Consider an arbitrary vector $\mathbf{a}$ and resolve it into components in O and O':

$$\mathbf{a} = a_i\hat{\mathbf{i}} + a_j\hat{\mathbf{j}} + a_k\hat{\mathbf{k}} = a'_i\hat{\mathbf{i}}' + a'_j\hat{\mathbf{j}}' + a'_k\hat{\mathbf{k}}'.$$

Differentiating with respect to time gives:

$$\begin{aligned}\frac{d\mathbf{a}}{dt} &= \frac{da_i}{dt}\hat{\mathbf{i}} + \frac{da_j}{dt}\hat{\mathbf{j}} + \frac{da_k}{dt}\hat{\mathbf{k}} \\ &= \frac{da'_i}{dt}\hat{\mathbf{i}}' + \frac{da'_j}{dt}\hat{\mathbf{j}}' + \frac{da'_k}{dt}\hat{\mathbf{k}}' + a'_i\boldsymbol{\omega}\times\hat{\mathbf{i}}' + a'_j\boldsymbol{\omega}\times\hat{\mathbf{j}}' + a'_k\boldsymbol{\omega}\times\hat{\mathbf{k}}'.\end{aligned}$$

At this point, we introduce some new notation. We normally use $\dot{\mathbf{a}}$ and $d\mathbf{a}/dt$ interchangeably. Let us now adopt the convention that

$$\dot{\mathbf{a}} \equiv \frac{da'_i}{dt}\hat{\mathbf{i}}' + \frac{da'_j}{dt}\hat{\mathbf{j}}' + \frac{da'_k}{dt}\hat{\mathbf{k}}',$$

which means that you differentiate the *components* of $\mathbf{a}$ but not the basis vectors, even if the basis vectors are time dependent. In other words, $\dot{\mathbf{a}}$ is the *rate of change of* $\mathbf{a}$ *measured in the rotating frame*. The *total* rate of change of $\mathbf{a}$ is then:

$$\boxed{\frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} + \boldsymbol{\omega}\times\mathbf{a}}.$$

There is one term for the rate of change with respect to the rotating axes and a second term arising from the rotation of the axes themselves.

4.2 Equation of Motion in a Rotating Frame

We can use the result we just derived to work out the equation of motion for a particle when its coordinates are measured in a frame rotating at *constant* angular velocity $\boldsymbol{\omega}$. Let $\mathbf{a}$ be a position vector $\mathbf{r}$. Differentiating once:

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}.$$

Differentiating again:

$$\begin{aligned} \frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \ddot{\mathbf{r}} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \end{aligned}$$

Newton's law of motion is $\mathbf{F}_{\text{tot}} = m\,d^2\mathbf{r}/dt^2$, where $\mathbf{F}_{\text{tot}}$ is the total force acting, so the equation of motion in the rotating frame becomes:

$$\boxed{m\ddot{\mathbf{r}} = \mathbf{F}_{\text{tot}} - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}.$$

The last two terms on the right hand side are *apparent* (or *inertial* or *fictitious*) forces, arising because we are measuring positions with respect to axes which are themselves rotating (i.e. accelerating).

4.3 Motion Near the Earth's Surface

Assume that the Earth is spherically symmetric so that the weight of an object is a vector directed towards the Earth's centre. Pick an inertial frame O with origin at the Earth's centre, together with a frame O' also with origin at the Earth's centre, but rotating with the Earth at angular velocity $\boldsymbol{\omega}$. Write the total force on the particle as its weight $m\mathbf{g}$ plus any other external forces $\mathbf{F}$ ($\mathbf{F}_{\text{tot}} = \mathbf{F} + m\mathbf{g}$).

Let $\mathbf{R}$ be a vector from the centre of the Earth to some point on or near its surface, as shown in figure 4.1, and let $\mathbf{x}$ be the displacement of the particle relative to this point. This says that the position vector in O' can be written as

$$\mathbf{r} = \mathbf{R} + \mathbf{x}.$$

Since $\mathbf{R}$ is fixed in O', $\dot{\mathbf{R}} = 0$ and $\ddot{\mathbf{R}} = 0$, and the equation of motion becomes:

$$m\ddot{\mathbf{x}} = \mathbf{F} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \dot{\mathbf{x}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{R} + \mathbf{x}]).$$

We will now drop all terms of order x/R or smaller. Even if x is 10 km, this ratio is $10\,\text{km}/6400\,\text{km} \approx 1.6 \times 10^{-3}$. With this approximation:

1. $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{R} + \mathbf{x}]) \longrightarrow \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$ (If R was not so large we would normally drop this $O(\omega^2)$ term),

2. the term involving $\mathbf{g}$ simplifies,

$$\mathbf{g} = -\frac{GM}{|\mathbf{R} + \mathbf{x}|^3}(\mathbf{R} + \mathbf{x}) \longrightarrow -\frac{GM}{R^3}\mathbf{R} = -g\frac{\mathbf{R}}{R}.$$

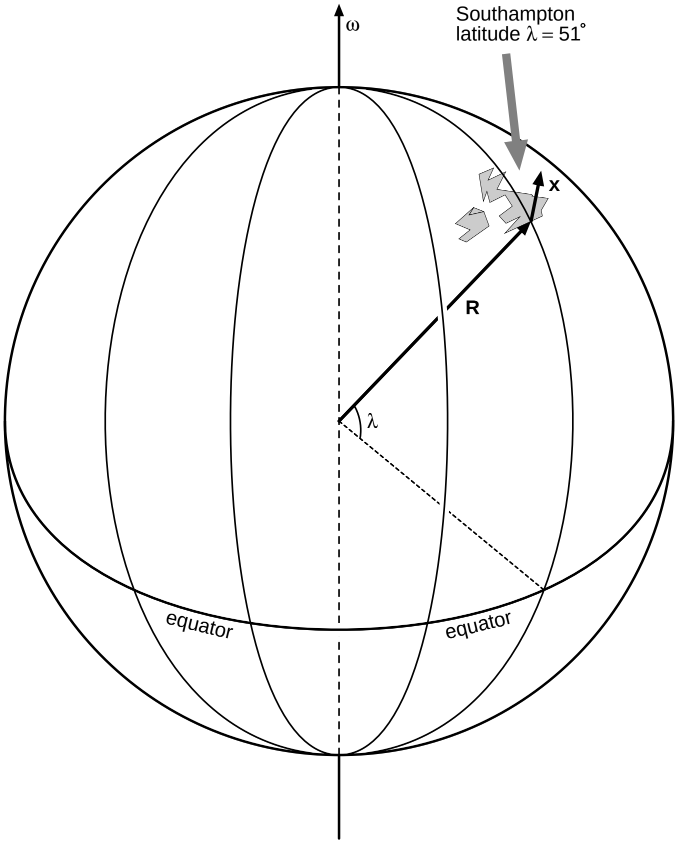


Figure 4.1 Motion near the surface of the Earth. Displacement **x** measured from tip of a (rotating) vector **R** from the Earth's centre to a point on or near its surface.

The approximate equation of motion becomes,

$$\boxed{m\ddot{\mathbf{x}} = \mathbf{F} + m\mathbf{g}^* - 2m\boldsymbol{\omega}\times\dot{\mathbf{x}}},$$

where we have defined the *apparent gravity*,

$$\boxed{\mathbf{g}^* = -g\frac{\mathbf{R}}{R} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})}.$$

We will take the latitude to be λ, as shown in figure 4.1 (note that latitude is zero at the equator).

4.3.1 Apparent Gravity

The apparent gravity $\mathbf{g}^*$ defines a local apparent vertical direction. It is what is measured by hanging a mass from a spring so that the mass is stationary in the rotating frame fixed to the Earth, and $\dot{\mathbf{x}} = 0$, $\ddot{\mathbf{x}} = 0$. We can easily work out the small deflection angle α between the apparent vertical and the true vertical defined by line to the Earth's centre. The situation is illustrated in figure 4.2.

The magnitude of the centrifugal term is,

$$|-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})| = \omega^2 R\cos\lambda.$$

Applying the cosine rule to the right hand triangle in figure 4.2 gives,

$$g^{*2} = g^2 + (\omega^2 R\cos\lambda)^2 - 2g\omega^2 R\cos^2\lambda,$$

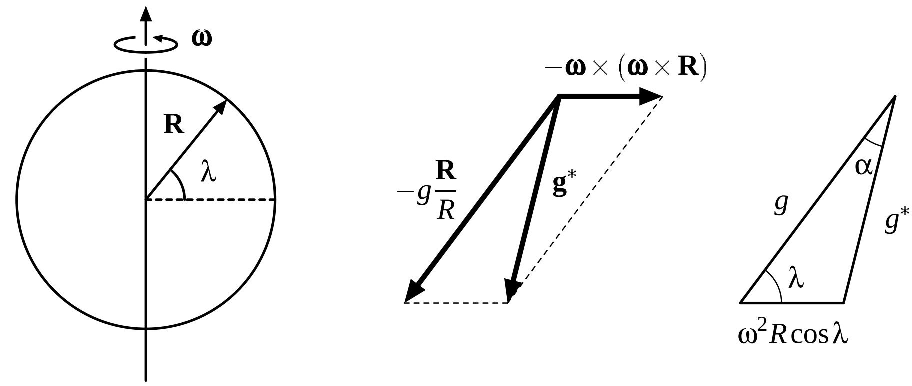


Figure 4.2 Determining the deflection angle between true and apparent verticals on the Earth's surface.

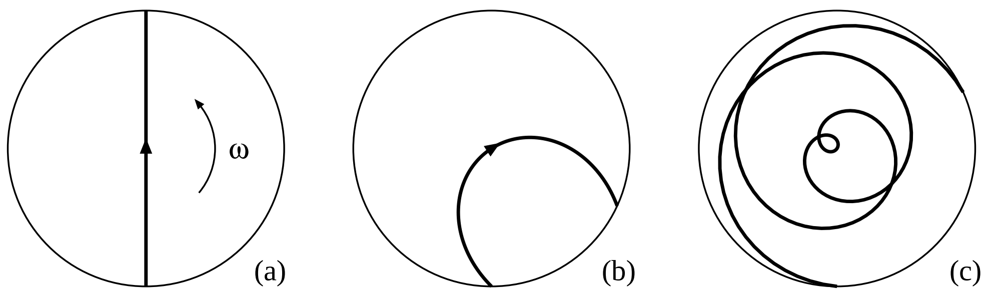


Figure 4.3 Particle moving across a rotating disc: seen from (a) an inertial frame, (b) a frame rotating with the disc, (c) a frame rotating with the disc when $\omega a/v$ is large, where v is the particle's speed in the inertial frame and a is the disc's radius.

which tells us that $g^* = g + O(\omega^2)$. Applying the sine rule to the same triangle gives,

$$\frac{\sin\alpha}{\omega^2 R\cos\lambda} = \frac{\sin\lambda}{g^*}.$$

Since α is small, we approximate $\sin\alpha \approx \alpha$, and to order ω^2 we can replace g^* by g, to find:

$$\boxed{\alpha = \frac{\omega^2 R}{g}\sin\lambda\cos\lambda}.$$

This tells us that the deflection vanishes at the equator and the poles, and is maximal at latitude $45°$. The size of the deflection is governed by

$$\frac{\omega^2 R}{g} = \frac{3.4\,\mathrm{cm\,s^{-2}}}{g} = 0.35\%.$$

At Southampton, $\lambda = 51°$, we find $\alpha = 1.7\times 10^{-3}\,\mathrm{rad} = 0°6'$.

4.3.2 Coriolis Force

The Coriolis "force" (in quotation marks because it's a fictitious or inertial force associated with our use of an accelerated frame) is the term

$$-2m\boldsymbol{\omega}\times\dot{\mathbf{x}}$$

in the equation of motion. You see that it acts at right angles to the direction of motion, and is proportional to the speed. To understand the physical origin of this

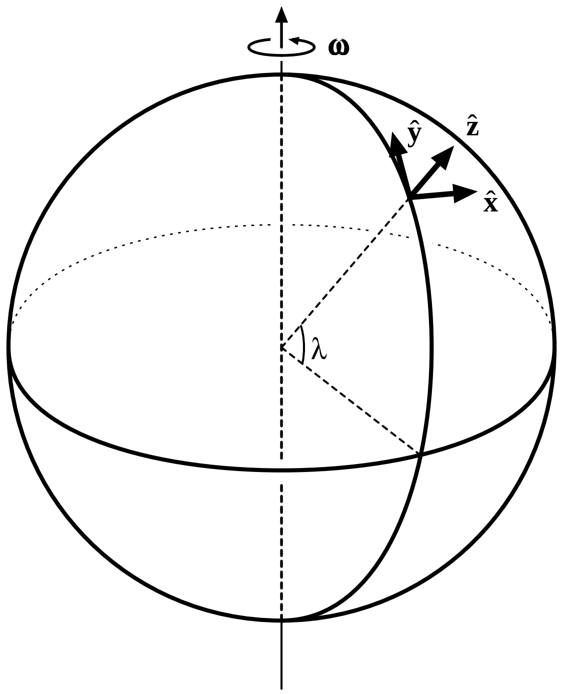


Figure 4.4 Coordinate system on the Earth's surface.

force, it may be helpful to consider a particle moving diametrically across a smooth flat rotating disc with no forces acting horizontally. An observer in an inertial frame (watching the disc from above) will simply see the particle move in a straight line at constant speed, as in figure 4.3(a). However, an observer rotating with the disc will see the particle follow a curved track as in figure 4.3(b). If the observer does not realise that the disc is rotating they will conclude that some force acts on the particle at right angles to its velocity: this is the Coriolis force (in this example, the rotating observer also sees the effect of the apparent force $m\omega^2\mathbf{x}$ acting radially outwards). As the rotation rate, ω, gets large, the path seen by the rotating observer can get quite complicated, figure 4.3(c).

To study the Coriolis force quantitatively, it is helpful to choose a convenient set of axes on the Earth's surface. This is done as follows, and is illustrated in figure 4.4. We choose $\hat{\mathbf{z}}$ along the apparent upward vertical (parallel to $-\mathbf{g}^*$), and take $\hat{\mathbf{x}}$ pointing to the East. The third unit vector $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ therefore points North. Using this coordinate system, the equations of motion are:

$$\begin{aligned} m\ddot{x} &= F_x - 2m\omega\left(\dot{z}\cos\lambda - \dot{y}\sin\lambda\right), \\ m\ddot{y} &= F_y - 2m\omega\dot{x}\sin\lambda, \\ m\ddot{z} &= F_z - mg^* + 2m\omega\dot{x}\cos\lambda. \end{aligned} \tag{4.1}$$

4.3.3 Free Fall — Effects of Coriolis Term

For a particle in free fall, the non-gravitational force $\mathbf{F}$ disappears from the equation of motion, which becomes,

$$\ddot{\mathbf{x}} = \mathbf{g}^* - 2\boldsymbol{\omega} \times \dot{\mathbf{x}}.$$

We will work to $O(\omega)$ in this section, so we can approximate $\mathbf{g}^*$ by $\mathbf{g}$.

We could investigate this using the coordinate form of the equation of motion given in equation (4.1). However, in this case, we can proceed vectorially and solve all three coordinate equations at the same time.

The equation of motion can be integrated once with respect to time, with the initial conditions $\mathbf{x} = \mathbf{a}$ and $\dot{\mathbf{x}} = \mathbf{v}$ at $t = 0$, corresponding to a particle projected with velocity $\mathbf{v}$ from point $\mathbf{a}$. This gives,

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{g}t - 2\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a}).$$

Since we are ignoring terms of $O(\omega^2)$, we can substitute the zeroth order solution, $\mathbf{x} = \mathbf{a} + \mathbf{v}t + \mathbf{g}t^2/2$ in the cross product term, giving,

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{g}t - 2\boldsymbol{\omega} \times \left(\mathbf{v}t + \frac{1}{2}\mathbf{g}t^2\right).$$

This can be integrated once more, using the same initial conditions, $\mathbf{x} = \mathbf{a}$ and $\dot{\mathbf{x}} = \mathbf{v}$ at $t = 0$, to give:

$$\boxed{\mathbf{x} = \mathbf{a} + \mathbf{v}t + \frac{1}{2}\mathbf{g}t^2 - \boldsymbol{\omega} \times \left(\mathbf{v}t^2 + \frac{1}{3}\mathbf{g}t^3\right)}.$$

Now that we have our solution, we can express it in terms of our choice of coordinates in figure 4.4. We will consider two cases: a particle dropped from a tower and a shell fired from a cannon.

Particle dropped from a tower Consider a particle dropped from rest from a vertical tower of height h. Writing a vector as a column of its components along our choice of axes, this says that the initial conditions are,

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}.$$

Using $\boldsymbol{\omega} \times \mathbf{g} = -\omega g \cos\lambda\, \hat{\mathbf{x}}$, we find that the components, x, y and z of $\mathbf{x}$ are:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \frac{1}{2}gt^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3}\omega g t^3 \cos\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The particle hits the ground when $z = 0$ at $t = \sqrt{2h/g}$. For this t, the x component of the particle's position is

$$\frac{1}{3}\omega\cos\lambda \left(\frac{8h^3}{g}\right)^{1/2}.$$

This says that the particle strikes the ground a little to the East of the base of the tower.

Two views of this are shown in figure 4.5. On the left is the view from a non-inertial frame fixed to the rotating Earth: the particle lands a little to the East of the base of the tower. On the right is a view from an inertial frame, where the Earth and tower are spinning beneath the observer. Now the particle is seen to be projected from the top of the tower. Because the particle is acted upon by the Earth's gravitational attraction, a central force, its angular momentum around the Earth's rotation axis is constant. As the particle falls, it gets closer to the axis, so its angular velocity must increase to keep the angular momentum constant. Therefore, the particle is again seen to get slightly ahead of the tower as it falls.

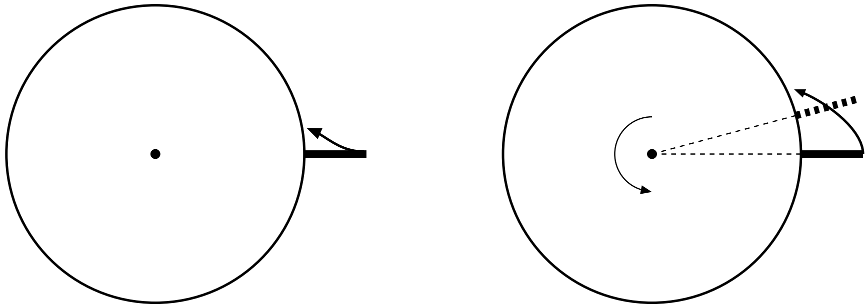


Figure 4.5 Two views of a particle dropped from the top of a tall tower fixed to the rotating Earth. On the left, as seen in a rotating frame fixed to the Earth, and on the right as seen in an inertial frame in which the Earth spins on its axis.

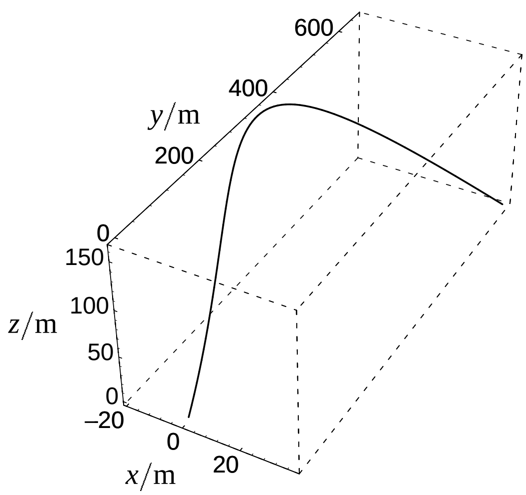


Figure 4.6 Deflection of a cannon shell by Coriolis force when viewed from non-inertial coordinates rotating with the Earth. A shell is fired at elevation angle $\pi/4$ with speed $80\,\mathrm{m\,s^{-1}}$ at latitude 24° in the Northern hemisphere. The Earth's angular velocity is set to $\omega = 0.05\,\mathrm{rad\,s^{-1}}$ to exaggerate the effect.

Shell fired from a cannon A shell is fired due North with speed v from a cannon, with elevation angle $\pi/4$. The initial conditions, taking the origin at the cannon, are now,

$$\mathbf{v} = \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

and the cross product of $\boldsymbol{\omega}$ with the initial velocity is,

$$\boldsymbol{\omega}\times\mathbf{v} = \frac{\omega v}{\sqrt{2}}(\cos\lambda - \sin\lambda)\,\hat{\mathbf{x}}.$$

Substituting in our solution we get:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{vt}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{2}gt^2\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{1}{3}\omega g t^3\cos\lambda\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{\omega v t^2}{\sqrt{2}}(\cos\lambda - \sin\lambda)\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

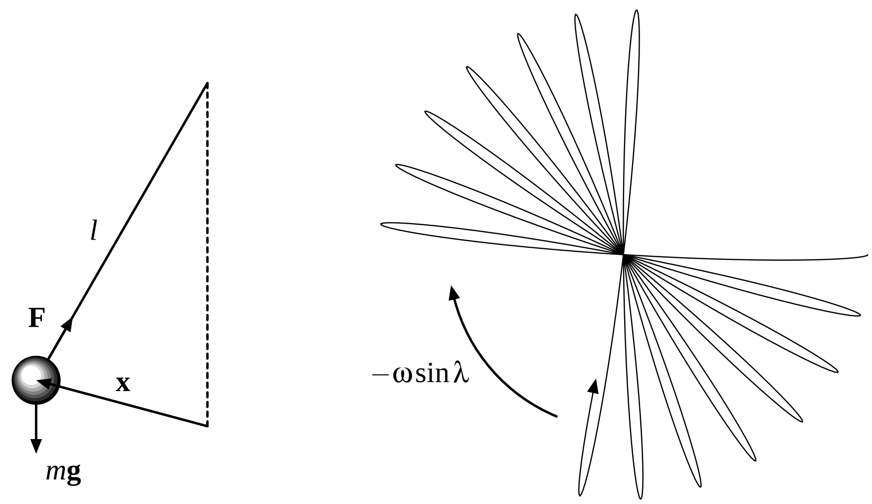


Figure 4.7 Foucault's pendulum and much exaggerated view of the path of the bob. The plane of oscillation rotates with angular velocity $-\omega \sin\lambda$, clockwise when seen from above.

Looking at the z component of this result shows that $z = vt/\sqrt{2} - gt^2/2$, so impact occurs at $t = \sqrt{2}v/g$. The Eastward deflection at impact is then found to be:

$$\frac{\sqrt{2}\omega v^3}{3g^2}(3\sin\lambda - \cos\lambda).$$

If $3\sin\lambda > \cos\lambda$ then the deflection at impact will be to the East. This occurs for $\lambda > \tan^{-1}(1/3) = 18.4°$, roughly the latitude of Mexico City or Bombay.

The Eastward deflection is the sum of a positive cubic term in the time t plus a quadratic term in t which is positive for $\lambda > 45°$. So, at Southampton, $\lambda = 51°$, the deflection is Eastward throughout the trajectory, but at latitudes below 45°, the deflection is initially to the West and then changes to the East. Figure 4.6 shows the trajectory up to the impact time, for $\lambda = 24°$, with an initial speed $v = 80\,\mathrm{m\,s^{-1}}$, but using a ridiculously large value, $\omega = 0.05\,\mathrm{rad\,s^{-1}}$, for the Earth's angular velocity to magnify the effect. This value of ω is about 700 times larger than the true value of about $7.3 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$. If the angular velocity were really as large as $0.05\,\mathrm{rad\,s^{-1}}$, we wouldn't be justified in using our small-ω approximation.

4.3.4 Foucault's Pendulum

If you were to set up a pendulum at the North pole and start it swinging in a plane (as viewed from an inertial frame — one not attached to the Earth), then clearly, according to an observer standing on the Earth, the plane of oscillation would rotate backwards at angular velocity $-\omega$.

At lower latitudes, the phenomenon persists, but gets more and more diluted until it vanishes at the equator. In fact, at latitude λ the plane of oscillation rotates at angular velocity $-\omega \sin\lambda$. This is illustrated, in a very exaggerated fashion, in figure 4.7. At Southampton, latitude 51°, the plane rotates about 10° in one hour. The effect was first demonstrated by Jean Foucault in Paris in 1851[1]. In practice, it is quite hard to start the pendulum with the correct initial conditions: the bob often ends up with a circular or elliptical path where the Foucault rotation is much harder to detect.

[1] For background, see the article *Léon Foucault*, Scientific American (July 1998) pp52–59

We will now derive the result for the rotation of the plane of oscillation. We make our standard choice of coordinates, shown in figure 4.4, with the z-axis along the upward local vertical, $\hat{\mathbf{z}} = -\mathbf{g}^*/g^*$. We will work to first order in the Earth's angular velocity ω, so we will drop the star on g^*. The system we consider is a pendulum of length l, free to swing in any direction with the same period, as illustrated in figure 4.7. The pendulum should be long and heavy so that it will swing for a long time, a matter of hours, in spite of air resistance (which we will neglect).

Measuring the displacement $\mathbf{x}$ of the bob from the bottom of the swing, the equations of motion in our coordinate system are just those of equation (4.1), where $\mathbf{F}$ is the tension in the support cable. In the approximation of small oscillations, we can ignore all z terms compared to x and y. Then, $F_x \approx -mgx/l$ and $F_y \approx -mgy/l$. The x and y equations now become,

$$\begin{aligned} \ddot{x} &= -\omega_0^2 x + 2\omega \sin\lambda \dot{y}, \\ \ddot{y} &= -\omega_0^2 y - 2\omega \sin\lambda \dot{x}, \end{aligned}$$

where we have defined $\omega_0^2 \equiv g/l$, so that ω_0 is the natural angular frequency of the pendulum. To solve these equations, define the complex quantity $\alpha = x + iy$. It is easy to see that the two equations above combine into a single equation for α,

$$\ddot{\alpha} + 2i\omega \sin\lambda \dot{\alpha} + \omega_0^2 \alpha = 0.$$

Look for a solution of the form $\alpha = Ae^{ipt}$. Substituting this form shows that we have a solution provided,

$$\begin{aligned} p &= -\omega \sin\lambda \pm \sqrt{\omega_0^2 + \omega^2 \sin^2\lambda} \\ &\approx -\omega \sin\lambda \pm \omega_0, \end{aligned}$$

where we have used $\omega_0 \gg \omega \sin\lambda$. The general solution is therefore,

$$\alpha = \left(Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right) e^{-i(\omega \sin\lambda)\,t},$$

where A and B are complex constants. With appropriate initial conditions the solution can be given as,

$$\boxed{\alpha = ae^{-i(\omega \sin\lambda)\,t} \cos(\omega_0 t)}.$$

The $\cos(\omega_0 t)$ term describes the usual periodic swing of the pendulum and the $e^{-i(\omega \sin\lambda)\,t}$ term describes the rotation of the plane of oscillation with angular velocity $-\omega \sin\lambda$, as shown in figure 4.7.

Geometric Description* There is a nice geometric way to think about the Foucault Pendulum which allows you to work out the rotation rate without solving a differential equation[2].

Draw parallel lines on a disc and then cut out a segment and fold the remainder into a cone. Choose the disc radius so that the edge of the cone sits on the Earth's surface at latitude λ, with the surface of the cone tangential to the Earth's surface where it touches. Keep the cone fixed in space as the Earth turns beneath it. As the Earth turns, the plane of swing of the Foucault pendulum always remains parallel to the lines drawn on the cone's surface. The construction is shown in figure 4.8. If you

[2]See J B Hart, R E Miller and R L Mills, Am. J. Phys. **55** (1987) 67.

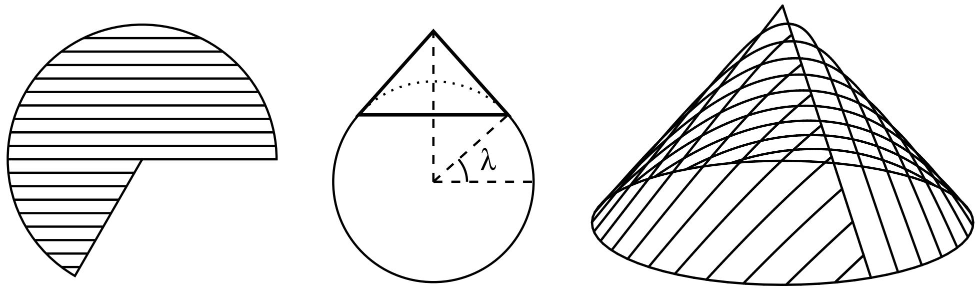


Figure 4.8 A geometric construction to find the rate of rotation of the plane of oscillation of a Foucault pendulum.

think about it, you should be able to figure out the rotation rate from the geometry (try it!).

This is an example of “parallel transport”: the plane of swing of the pendulum is parallel-transported as the Earth rotates. This concept is very important in differential geometry, which underlies general relativity.

5

Simple Harmonic Motion*

Note: this section is not part of the syllabus for PHYS2006. You should already be familiar with simple harmonic motion from your first year course PH115 *Oscillations and Waves*. This section is included for completeness and as a reminder.

5.1 Simple Harmonic Motion

This is one of the most important phenomena in physics: it applies to the description of small oscillations of any system about a position of stable equilibrium.

Work in one dimension, so that one coordinate describes the position of the system (e.g. the displacement from the equilibrium position of a spring, the angle of a pendulum from the vertical). Only conservative forces do work, so there is a potential $V(x)$. Choose coordinates so that $x=0$ is a position of stable equilibrium. This means

$$F(x{=}0) = 0, \qquad -\left.\frac{dV}{dx}\right|_0 = 0.$$

As long as x remains small, we can expand the potential:

$$V(x) = V(0) + xV'(0) + \frac{1}{2}x^2V''(0) + \cdots$$

However, $V'(0) = 0$, since $x = 0$ is a position of equilibrium, so the first derivative term vanishes. Letting $k = V''(0)$ (k is just the force constant for a spring force) and choosing our zero of potential energy so that $V(0) = 0$, we find:

$$V(x) = \frac{1}{2}kx^2 + \cdots.$$

The corresponding force is $F(x) = -kx$. We ignore the special case $k = 0$, when the expansion of V begins at higher order. If $k < 0$ then the equilibrium is unstable, and the system will move out of the region where our approximation is valid. Hence we will look at displacements around positions of stable equilibrium for which $k > 0$.

We define a *Simple Harmonic Oscillator* as a one-dimensional problem with:

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2x^2$$

where $k > 0$ and we have defined $\omega_0^2 = k/m$.

A mass oscillating on a Hooke's law spring is a simple harmonic oscillator. Small oscillations of a simple pendulum are simple harmonic.

5.1.1 General Solution

The equation of motion for the simple harmonic oscillator is

$$\ddot{x} + \omega_0^2 x = 0.$$

This is a second order homogeneous linear differential equation, meaning that the highest derivative appearing is a second order one, each term on the left contains exactly one power of x, $\dot{x}$ or $\ddot{x}$ (there is no $\dot{x}$ term in this case) and there is no term (a constant or a function of time) on the right.

Two independent solutions of this are $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$. The general solution is a linear combination of these, which can be written in several forms:

$$\begin{aligned} x &= A\cos(\omega_0 t) + B\sin(\omega_0 t) \\ &= C\cos(\omega_0 t + \delta) \\ &= D\sin(\omega_0 t + \varepsilon) \\ &= \mathrm{Re}\left(\alpha e^{i\omega_0 t}\right) \\ &= \mathrm{Im}\left(\beta e^{-i\omega_0 t}\right) \end{aligned}$$

where A, B, C, D, δ and ε are real constants, and α and β are complex constants. Use whichever solution is most convenient. We will often use the complex exponential forms, so we will need to remember that the physical solutions are found by taking the real or imaginary parts. Some terminology associated with the simple harmonic oscillator is:

angular frequency	ω_0
period	$T = 2\pi/\omega_0$
amplitude	$a = \lvert C\rvert = \lvert D\rvert = \sqrt{A^2 + B^2} = \lvert\alpha\rvert = \lvert\beta\rvert$

The arguments of the sine or cosine in $\cos(\omega_0 t + \delta)$ and $\sin(\omega_0 t + \varepsilon)$ are called the phase. The period of a simple harmonic oscillator is independent of the amplitude: this is a special property, not true for oscillators in general.

5.2 Damped Harmonic Motion

We'll assume that a damping force proportional to speed is present,

$$F_{\text{damping}} = -2m\gamma\dot{x}.$$

This equation defines γ (note that in defining γ we have pulled out one factor of m for convenience: γ could still itself depend on m). *Warning*: many authors use $\gamma/2$ in place of γ.

In general, the damping can be some power series in $\dot{x}$. We approximate by keeping the linear term only. In practice, this turns out to work well: the viscously damped harmonic oscillator is a very useful model for all sorts of physical systems.

The equation of motion has become:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0.$$

This is still a linear, homogeneous second order differential equation. We try a solution of the form

$$x = Ae^{\Omega t}$$

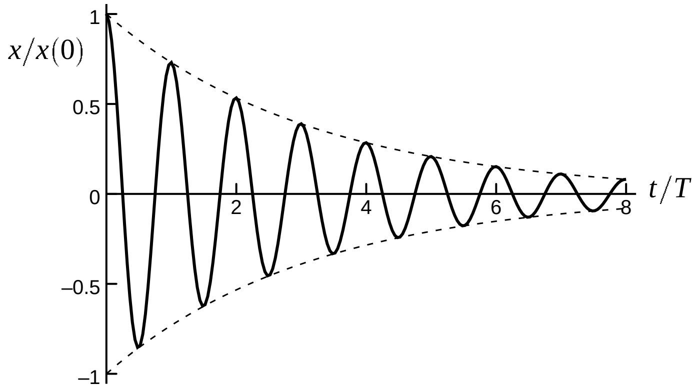


Figure 5.1 Amplitude as a function of time for a lightly damped harmonic oscillator. The time is measured in units of the "period" $T = 2\pi/\omega$. The dashed lines show the exponentially damped envelope of the oscillatory motion.

where A and Ω may be complex (and we take the real part at the end). We will do this same trick of using a complex exponential many times more. Substituting our trial solution gives:

$$(\Omega^2 + 2\gamma\Omega + \omega_0^2)Ae^{\Omega t} = 0.$$

Since the equation is linear, A is arbitrary, and we want it non zero in order to have a non-trivial solution. The factor in brackets then gives a quadratic for Ω: the two roots of this will provide us with our two independent solutions.

5.2.1 Small Damping: $\gamma^2 < \omega_0^2$

The roots of the quadratic are

$$\Omega = -\gamma \pm i\omega, \qquad \text{where} \qquad \omega = \sqrt{\omega_0^2 - \gamma^2}.$$

A solution may be written $x = \mathrm{Re}\,(A_1 e^{i\omega t} + A_2 e^{-i\omega t})e^{-\gamma t}$, which can be reexpressed as:

$$x = Be^{-\gamma t}\cos(\omega t + \delta).$$

This describes an oscillation with "frequency" $\omega = \sqrt{\omega_0^2 - \gamma^2}$ and exponentially decaying "amplitude" $Ae^{-\gamma t}$, as illustrated in figure 5.1. The quotes are here because the motion is no longer periodic, so there is not really a frequency. However, you could use the time between the system crossing $x = 0$ in the *same* direction as a measure of a "period", since this time is $2\pi/\omega$. If the damping is truly small, then the oscillations will appear to have amplitude $Ae^{-\gamma t}$ if you watch them for a short interval around time t.

In one "period", $T = 2\pi/\omega$ of a lightly damped oscillator's motion, the fractional energy loss is found by comparing the total energy at the start of the period and at the end. For any time, t, the fractional loss is given by

$$\frac{\Delta E}{E} = \frac{E(t) - E(t+T)}{E(t)} = 1 - e^{-2\gamma T}.$$

When the damping is very small, $\gamma/\omega_0 \ll 1$, we have $\omega \approx \omega_0$ and then

$$\frac{\Delta E}{E} \approx 2\pi\frac{2\gamma}{\omega_0} \equiv \frac{2\pi}{Q},$$

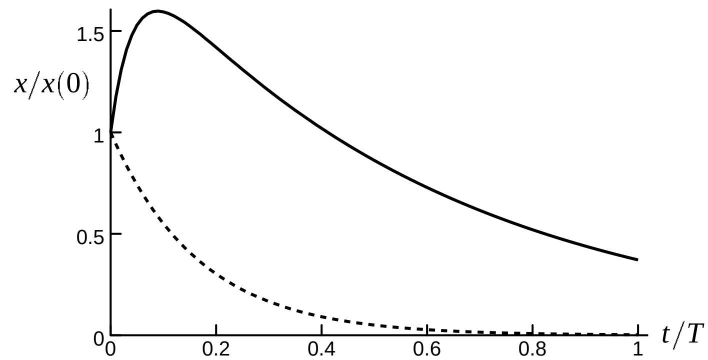


Figure 5.2 Amplitude as a function of time for heavily damped (solid curve) and critically damped (dashed curve) harmonic oscillators. The time is measured in units of the natural period $T = 2\pi/\omega_0$ of the oscillator when the damping is switched off.

which defines the *quality factor* Q. *Warning:* definitions of Q vary from author to author.

5.2.2 Large Damping: $\gamma^2 > \omega_0^2$

The roots of the quadratic are

$$\Omega = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

and a solution may be written

$$x = Ae^{-(\gamma+\sqrt{\gamma^2-\omega_0^2})t} + Be^{-(\gamma-\sqrt{\gamma^2-\omega_0^2})t}.$$

This is a sum of two exponentials, both decaying with time, illustrated by the solid curve in figure 5.2. The "*B*" exponential falls more slowly, so it dominates at large times. This case is sometimes referred to as "overdamped".

5.2.3 Critical Damping: $\gamma^2 = \omega_0^2$

In this special case the solutions for Ω are degenerate (the roots of the quadratic coincide). It looks as though there is just one solution. However, a second order differential equation *must* have two independent solutions. You can check by differentiating that the second solution in this case is

$$x = Bte^{-\gamma t},$$

so that the general solution becomes:

$$x = (A + Bt)e^{-\gamma t}.$$

The critically damped solution is illustrated by the dashed curve in figure 5.2. Critical damping is important: for example a measuring instrument should be critically damped so that the reading settles down as fast as possible without the response time being too slow.

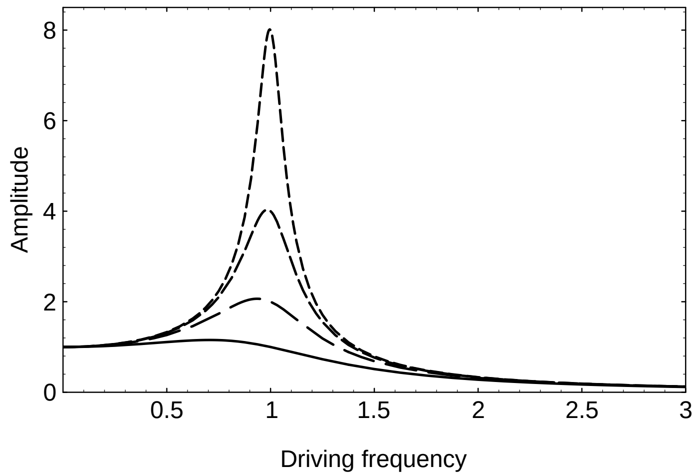


Figure 5.3 Amplitude of forced harmonic oscillator as a function of driving frequency (in units of natural frequency)

5.3 Driven damped harmonic oscillator

The equation of motion for a damped harmonic oscillator driven by an external force $F(t)$ is

$$m\ddot{x} + 2m\gamma\dot{x} + m\omega_0^2 x = F(t).$$

Consider the case of a periodic driving force,

$$F(t) = mf\cos(\omega t) = mf\mathrm{Re}\,(e^{i\omega t}),$$

and look for the *steady state* solution, when any *transient* damped solution has died away (the transients are solutions of the differential equation without the driving term $F(t)$, that is, a free damped oscillator). Look for a complex z which solves

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = fe^{i\omega t},$$

and take the real part of z at the end. Try a trial solution $z = Ae^{i\omega t}$: the idea is that after a long time we expect the system to be oscillating with the same frequency as the driving force. More technically, the full solution of the differential equation is the sum of the solution we are about to find plus *any* solution of the undriven equation (without the $fe^{i\omega t}$ term). Because of the damping, the solution in the undriven case decays exponentially with time: we are interested in what happens after a long time when this *transient* solution has died out.

Returning to our trial solution, $z = Ae^{i\omega t}$ solves the equation if

$$(-\omega^2 + 2i\gamma\omega + \omega_0^2)Ae^{i\omega t} = fe^{i\omega t}.$$

Cancelling the $e^{i\omega t}$ from both sides and solving for A gives

$$A = \frac{f}{-\omega^2 + 2i\gamma\omega + \omega_0^2}.$$

Writing $A = |A|e^{-i\delta}$, we find that the oscillation amplitude $|A|$ and phase lag δ are given by,

$$|A| = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

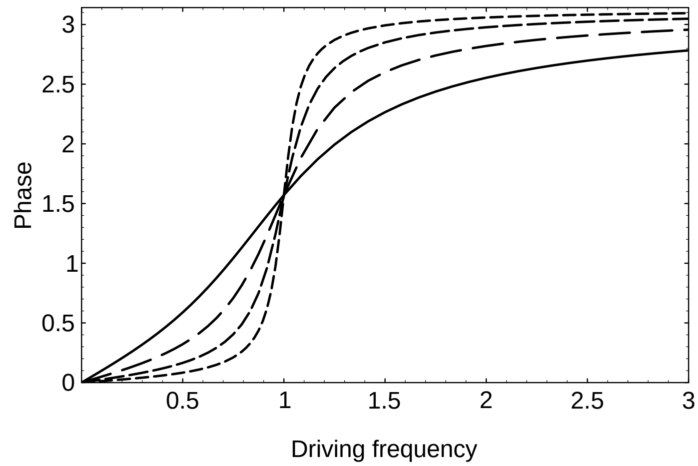


Figure 5.4 Phase lag of forced harmonic oscillator as a function of driving frequency (in units of natural frequency)

and

$$\tan\delta = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}.$$

In figures 5.3 and 5.4 we plot the amplitude (actually $\omega_0^2|A|/f$) and the phase δ as functions of ω/ω_0, for four different values of the *quality factor* $Q = \omega_0/2\gamma$. The quality factor tells you about the ratio of the energy stored in the oscillator to the energy loss per cycle. As you move from solid to finer and finer dashed lines the Q values are 1, 2, 4 and 8 respectively.

6

Coupled Oscillators

In what follows, I will assume you are familiar with the simple harmonic oscillator and, in particular, the complex exponential method for finding solutions of the oscillator equation of motion. If necessary, consult the revision section on Simple Harmonic Motion in chapter 5.

6.1 Time Translation Invariance

Before looking at coupled oscillators, I want to remind you how time translation invariance leads us to use (complex) exponential time dependence in our trial solutions. Later, we will see that spatial translation invariance leads to exponential forms for the spatial parts of our solutions as well.

To examine the implication of time translation invariance, it's enough to consider a single damped harmonic oscillator, with equation of motion,

$$m\ddot{x} = -2m\gamma\dot{x} - m\omega_0^2 x,$$

where the two terms on the right are the damping and restoring forces respectively. We can rearrange this to,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0.$$

To solve this equation, we used an ansatz (or guess) of the form

$$x = Ae^{\Omega t},$$

where A and Ω are in general complex (to get a physical solution you can use the real or imaginary parts of a complex solution). The reason that we could guess such a solution lies in time translation invariance.

What this invariance means is that we don't care about the origin of time. It doesn't matter what our clock read when we started observing the system. In the differential equation, this property appears because the time dependence enters only through time derivatives, *not* through the value of time itself. In terms of a solution $x(t)$, this means that:

$$\boxed{\text{if } x(t) \text{ is a solution, then so is } x(t+c) \text{ for any constant } c.}$$

The simplest possibility is that $x(t+c)$ is proportional to $x(t)$, with some proportionality constant $f(c)$, depending on c,

$$x(t+c) = f(c)x(t).$$

We can solve this equation by a simple trick. We differentiate with respect to c and then set $c = 0$ to obtain

$$\dot{x}(t) = \Omega x(t),$$

where Ω is just the value of $\dot{f}(0)$. The general solution of this linear first order differential equation is

$$x(t) = Ae^{\Omega t}.$$

We often talk about *complex* exponential forms because Ω must have a non-zero imaginary part if we want to get oscillatory solutions. In fact, from now on I will let $\Omega = i\omega$, so that ω is real for a purely oscillatory solution.

We can't just use *any* value we like for ω. The allowed values are determined by demanding that $Ae^{i\omega t}$ actually solves the equation of motion:

$$(-\omega^2 + 2i\gamma\omega + \omega_0^2)Ae^{i\omega t} = 0.$$

If we are to have a non-trivial solution, A should not vanish. The factor in parentheses must then vanish, giving a quadratic equation to determine ω. The two roots of the quadratic give us two independent solutions of the original second order differential equation.

6.2 Normal Modes

We want to generalise from a single oscillator to a set of oscillators which can affect each others' motion. That is to say, the oscillators are *coupled*.

If there are n oscillators with positions $x_i(t)$ for $i = 1, \ldots, n$, we will denote the "position" of the whole system by a vector $\mathbf{x}(t)$ of the individual locations:

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

The individual positions $x_i(t)$ might well be generalised coordinates rather than real physical positions.

The differential equations satisfied by the x_i will involve time dependence only through time derivatives, which means we can look for a time translation invariant solution, as described above. This means all the oscillators must have the same complex exponential time dependence, $e^{i\omega t}$, where ω is real for a purely oscillatory motion. The solution then takes the form,

$$\mathbf{x}(t) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} e^{i\omega t},$$

where the A_i are *constants*. This describes a situation where all the oscillators move with the *same frequency*, but, in general, different phases and amplitudes: the oscillators' displacements are in fixed ratios determined by the A_i. This kind of motion is called a *normal mode*. The *overall* normalisation is arbitrary (by linearity of the differential equation), which is to say that you can multiply all the A_i by the same constant and still have the same normal mode.

Our job is to discover which ω are allowed, and then determine the set of A_i corresponding to each allowed ω. We will find precisely the right number of normal modes to provide all the independent solutions of the set of differential equations. For n oscillators obeying second order coupled equations there are $2n$ independent solutions: we will find n coupled normal modes which will give us $2n$ real solutions when we take the real and imaginary parts.

Once we have found all the normal modes, we can construct *any* possible motion of the system as a linear combination of the normal modes. Compare this with Fourier analysis, where any periodic function can be expanded as a series of sines and cosines.

6.3 Coupled Oscillators

Take a set of coupled oscillators described by a set of coordinates $q_1, \ldots, q_n$. In general the potential $V(q)$ will be a complicated function which couples all of these oscillators together. Consider *small* oscillations about a position of stable equilibrium, which (by redefining our coordinates if necessary) we can take to occur when $q_i = 0$ for $i = 1, \ldots, n$. Expanding the potential in a Taylor series about this point, we find,

$$V(q) = V(0) + \sum_i \left.\frac{\partial V}{\partial q_i}\right|_0 q_i + \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0 q_i q_j + \cdots.$$

By adding an overall constant to V we can choose $V(0) = 0$. Since we are at a position of equilibrium, all the first derivative terms vanish. So the first terms that contribute are the second derivative ones. We define,

$$K_{ij} \equiv \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0,$$

and drop all the remaining terms in the expansion. Note that K_{ij} is a constant symmetric (why?) $n \times n$ matrix. The corresponding force is thus

$$F_i = -\frac{\partial V}{\partial q_i} = -\sum_j K_{ij} q_j$$

and thus the equations of motion are

$$M_i \ddot{q}_i = -\sum_j K_{ij} q_j,$$

for $i = 1, \ldots, n$. Here the M_i are the masses of the oscillators, and K is a matrix of ‘spring constants’. Indeed for a system of masses connected by springs, with each mass moving in the same single dimension, the coordinates can be taken as the real position coordinates, and then M is a (diagonal in this case) matrix of masses, while K is a matrix determined by the spring constants. Be aware however, that coupled oscillator equations occur more generally (for example in electrical circuits) where the q_is need not be actual coordinates but more general parameters describing the system (known as generalised coordinates) and in this case M and K play similar rôles even if they do not in actuality correspond to masses and spring constants.

To simplify the notation, we will write the equations of motion as a matrix equation. So we define,

$$\mathbf{M} = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_n \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{pmatrix}.$$

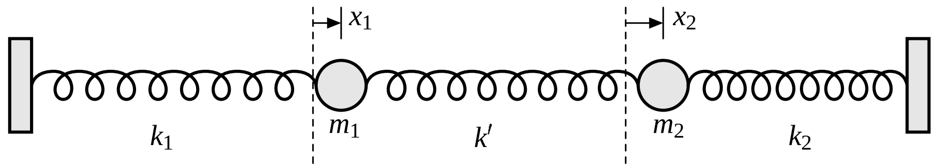


Figure 6.1 Two coupled harmonic oscillators. The vertical dashed lines mark the equilibrium positions of the two masses.

Likewise, let $\mathbf{q}$ and $\ddot{\mathbf{q}}$ be column vectors,

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}, \qquad \ddot{\mathbf{q}} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{pmatrix}.$$

With this notation, the equation of motion is,

$$\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}, \qquad \text{or} \qquad \ddot{\mathbf{q}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{q},$$

where $\mathbf{M}^{-1}$ is the inverse of $\mathbf{M}$.

Now look for a normal mode solution, $\mathbf{q} = \mathbf{A}e^{i\omega t}$, where $\mathbf{A}$ is a column vector. We have $\ddot{\mathbf{q}} = -\omega^2\mathbf{q}$, and cancelling $e^{i\omega t}$ factors, gives finally,

$$\boxed{\mathbf{M}^{-1}\mathbf{K}\mathbf{A} = \omega^2\mathbf{A}}.$$

This is now an *eigenvalue equation*. The squares of the normal mode freqencies are the *eigenvalues* of $\mathbf{M}^{-1}\mathbf{K}$, with the column vectors $\mathbf{A}$ as the corresponding *eigenvectors*.

6.4 Example: Masses and Springs

As a simple example, let's look at the system shown in figure 6.1, comprising two masses m_1 and m_2 constrained to move along a straight line. The masses are joined by a spring with force constant k', and m_1 (m_2) is joined to a fixed wall by a spring with force constant k_1 (k_2). Assume that the equilibrium position of the system has each spring unstretched, and use the displacements x_1 and x_2 of the two masses away from their equilibrium positions as coordinates. The force on mass m_1 is then

$$F_1 = -k_1x_1 - k'(x_1 - x_2)$$

and on mass m_2

$$F_2 = -k_2x_2 - k'(x_2 - x_1).$$

(Note that these follow from a potential of form $V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k'(x_2 - x_1)^2 + \frac{1}{2}k_2x_2^2$.) You can check that Newton's 2nd law thus implies, in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k' & -k' \\ -k' & k_2 + k' \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The eigenvalue equation we have to solve is:

$$\begin{pmatrix} (k_1 + k')/m_1 & -k'/m_1 \\ -k'/m_2 & (k_2 + k')/m_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \omega^2\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

Now specialise to a case where $m_1 = m$, $m_2 = 2m$, $k_1 = k$, $k_2 = 2k$ and $k' = 2k$. The eigenvalue equation becomes,

$$\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{m}{k}\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

or, setting $\lambda = m\omega^2/k$,

$$\begin{pmatrix} 3-\lambda & -2 \\ -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For there to be a solution, the determinant of the 2×2 matrix in the last equation must vanish. This gives a quadratic equation for λ,

$$\lambda^2 - 5\lambda + 4 = 0,$$

with roots $\lambda = 1$ and $\lambda = 4$. The corresponding eigenfrequencies are $\omega = \sqrt{k/m}$ and $\omega = 2\sqrt{k/m}$. For each eigenvalue, there is a corresponding eigenvector. With $\lambda = 1$ you find $A_2 = A_1$, and with $\lambda = 4$ you find $A_2 = -A_1/2$. Note that just the ratio of the two A_i is determined: you can multiply all the A_i by a constant and stay in the same normal mode. This means that we are free to normalise the eigenvectors as we choose. It is common to make them have unit modulus, in which case the eigenfrequencies and eigenvectors are:

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}}, & \mathbf{A} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \omega &= 2\sqrt{\frac{k}{m}}, & \mathbf{A} &= \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \end{aligned}$$

In the first normal mode, the two masses swing in phase with the same amplitude, and the middle spring remains unstretched. This could have been predicted: we have solved for a case where m_2 is twice the mass of m_1, and is attached to a wall by a spring with twice the force constant. Therefore, m_1 and m_2 would oscillate with the same frequency in the absence of the connecting spring.

In the second mode the two masses move out of phase with each other, and m_1 has twice the amplitude of m_2.

6.4.1 Weak Coupling and Beats

Now consider a case where the two masses are equal, $m_1 = m_2 = m$, and the two springs attaching the masses to the fixed walls are identical, $k_1 = k_2 = k$. From the symmetry of the setup, you expect one mode where the two masses swing in phase with the same amplitude, the central connecting spring remaining unstretched. In the second mode, the two masses again have the same amplitude, but swing out of phase, alternately approaching and receding from each other. This second mode will have a higher frequency (why?).

If the spring constant of the connecting spring is $k' = \varepsilon k$, you should check that applying the solution method worked through above gives the following eigenfrequencies and normal modes:

$$\begin{aligned} \omega_1 &= \sqrt{\frac{k}{m}}, & \mathbf{A}_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \omega_2 &= \sqrt{(1+2\varepsilon)\frac{k}{m}}, & \mathbf{A}_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned}$$

When the connecting spring has a very small force constant, $\varepsilon \ll 1$, so that the coupling is weak, the two normal modes have almost the same frequency. In this case it's possible to observe *beats* when a motion contains components from both normal modes. For example, suppose you start the system from rest by holding the left hand mass with a small displacement to the right, say d, keeping the right hand mass in its equilibrium position, and then letting go.

A general solution for the motion has the form,

$$\mathbf{x}(t) = c_1\mathbf{A}_1\cos(\omega_1 t) + c_2\mathbf{A}_2\cos(\omega_2 t) + c_3\mathbf{A}_1\sin(\omega_1 t) + c_4\mathbf{A}_2\sin(\omega_2 t).$$

Because the system starts from rest, you can immediately see (make sure you can!) that $c_3 = c_4 = 0$ in this case. Then the initial condition,

$$\mathbf{x}(0) = \begin{pmatrix} d \\ 0 \end{pmatrix},$$

gives,

$$\begin{pmatrix} d \\ 0 \end{pmatrix} = \frac{c_1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{c_2}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which is solved by $c_1 = c_2 = d/\sqrt{2}$. So, the motion is given by:

$$\begin{aligned} x_1(t) &= \frac{d}{2}(\cos(\omega_1 t) + \cos(\omega_2 t)), \\ x_2(t) &= \frac{d}{2}(\cos(\omega_1 t) - \cos(\omega_2 t)). \end{aligned}$$

We can rewrite the sum and difference of cosines as products, leaving:

$$\begin{aligned} x_1(t) &= d\cos\left(\frac{\omega_2 - \omega_1}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right), \\ x_2(t) &= d\sin\left(\frac{\omega_2 - \omega_1}{2}t\right)\sin\left(\frac{\omega_1 + \omega_2}{2}t\right). \end{aligned}$$

Now you can see that each of x_1 and x_2 has a "fast" oscillation at the average frequency $(\omega_1 + \omega_2)/2$, modulated by a "slow" amplitude variation at the difference frequency $(\omega_2 - \omega_1)/2$. The displacements show the contributions of the two normal modes beating together, as illustrated in figure 6.2.

You can easily demonstrate beats by tying a length of cotton between two chairs and hanging two keys from it by further equal-length threads. Each key is a simple pendulum and the suspension thread provides a weak coupling between them. Start the system by pulling one of the keys to one side, with the other hanging vertically, and releasing, so that you start with one key swinging from side to side and the other at rest. The swinging key gradually reduces its amplitude, and at the same time the other key begins to move. Eventually, the first key will momementarily stop swinging, whilst the second key has reached full amplitude. The process then continues, and the swinging motion transfers back and forth between the two keys.

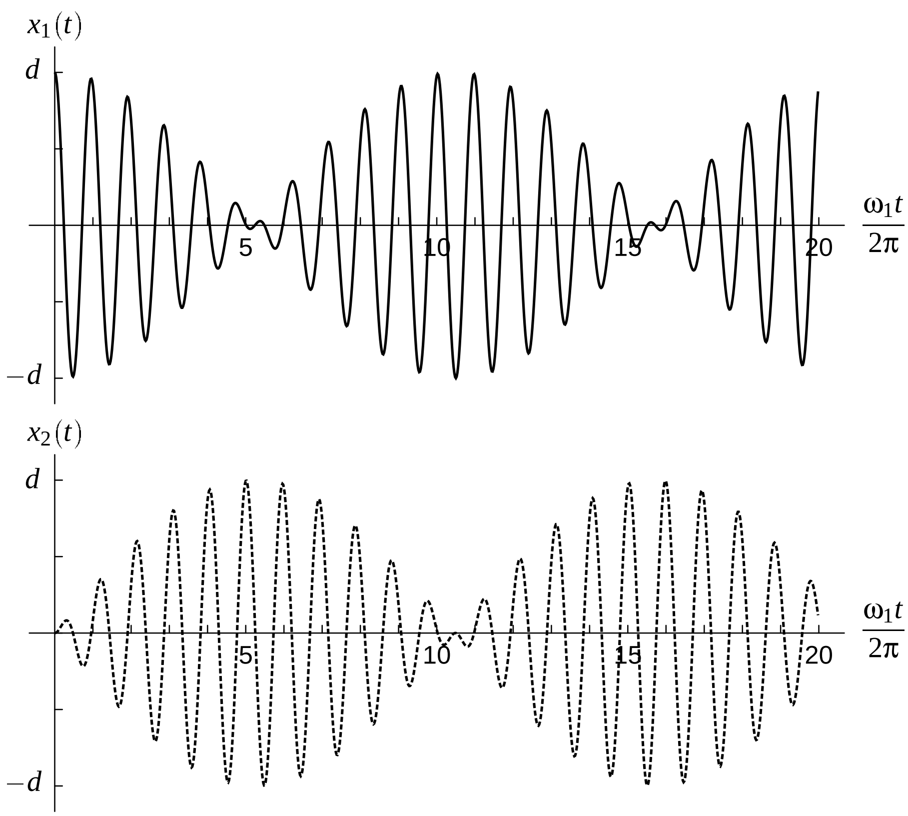


Figure 6.2 Displacements x_1 and x_2 as functions of time, starting with both masses at rest and $x_1(0) = d$, $x_2(0) = 0$. The displacement curve for x_2 is shown dashed. For this plot, the ratio ε of the spring force constants of the coupling (central) spring and either of the outer springs is 0.1. Time is plotted in units of the period of the lower frequency normal mode.

7

Normal Modes of a Beaded String

7.1 Equation of Motion

The system we will describe is a string stretched to tension T, carrying N beads, each of mass M, as shown in figure 7.1. The beads are equally spaced distance a apart, and the ends of the string are distance a from the first and last bead respectively. We will consider small transverse oscillations of the beads, with the ends of the string held in fixed positions.

If the displacement of the nth bead is u_n, we can work out its equation of motion by applying Newton's second law. Referring to the lower part of figure 7.1, we find:

$$M\ddot{u}_n = -T\,(\sin\psi + \sin\phi).$$

If the displacements are all small, then

$$\sin\psi \approx \frac{u_n - u_{n-1}}{a}, \qquad \text{and} \qquad \sin\phi \approx \frac{u_n - u_{n+1}}{a}.$$

Applying this approximation, the equations of motion are

$$\boxed{\ddot{u}_n = \frac{T}{Ma}(u_{n-1} - 2u_n + u_{n+1})}.$$

You get the same equation for longitudinal oscillations of a one-dimensional line of masses connected by identical springs, with C/M replacing T/Ma, where C is the spring constant of each spring.

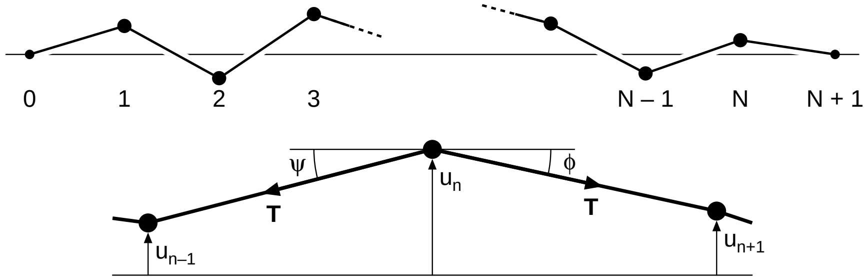


Figure 7.1 Transverse oscillations of a beaded string.

We can incorporate the boundary conditions, that the ends of the string are fixed, by requiring

$$u_0 = 0, \qquad u_{N+1} = 0.$$

You should convince yourself that these conditions give the right equations of motion for the first and Nth beads.

7.2 Normal Modes

We would like to find the normal modes of the beaded string. These are motions where all the beads oscillate with the same angular frequency ω:

$$u_n = A_n e^{i\omega t},$$

for some set of coefficients A_n. Substituting in the equation of motion gives,

$$\omega^2 A_n = \frac{T}{Ma}\left(-A_{n-1} + 2A_n - A_{n+1}\right). \tag{7.1}$$

This is a *recurrence relation* for the A_n — it is a discrete form of a differential equation. The boundary conditions are now incorporated as,

$$A_0 = A_{N+1} = 0.$$

We could solve for the A_n by viewing the recurrence relation as a matrix equation determining the column vector of the A_n's, like we did for systems with one or two degrees of freedom. Alternatively, we could apply known methods of solving recurrence relations. Rather than do either of these things, we will use some physical insight, allowing us almost to write down the solution with little effort. There are two key points:

- Suppose we actually had an *infinite* line of beads on a string. The infinite system has a *translation invariance*. If you jump one step (or any integer number of steps) left or right, the system looks the same. This will make it easy to find the normal modes of the infinite system.

- Each bead is connected to its two nearest neighbours only: the interaction is *local*. In the equation of motion, u_n is affected only by u_{n-1}, u_{n+1} and u_n itself, so the nth bead's displacement is affected *only* by the displacements of its two neighbours. Thus, if you can find a combination of normal modes of the infinite system which satisfies $A_0 = A_{N+1} = 0$, then you'll have found a mode of the finite system. You don't care what A_{-1}, A_{N+2} and so on are doing.

To repeat, we will look for normal modes by finding modes for an infinite line of beads and then selecting particular combinations of modes to satisfy the boundary conditions that the ends of the finite string are fixed.

7.2.1 Infinite System: Translation Invariance

Suppose we have already found a mode for the infinite string, with some set of displacement amplitudes A_n.

Now shift the system one step to the left. The translation invariance tells us it looks the same. This means that if the A_n gave us a mode with frequency ω, the shifted A'_n should give another mode with the *same* ω. That is, the new set of amplitudes,

$$A'_n = A_{n+1},$$

also give a mode.

Now let's look for a translation invariant mode, which reproduces itself when we do the shift. Since a mode is arbitrary up to an overall scale, this means,

$$A'_n = A_{n+1} = hA_n,$$

for some constant h, so that the new amplitudes are proportional to the old ones. Applying the last relation repeatedly shows that,

$$A_n = h^n A_0,$$

where A_0 is arbitrary and sets the overall scale. Given this set of A_n, we can find the corresponding angular frequency ω by substituting in the equation of motion in the form it appeared in equation (7.1). We find,

$$\omega^2 h^n A_0 = \frac{T}{Ma}\left(-h^{n-1}A_0 + 2h^n A_0 - h^{n+1}A_0\right).$$

Cancelling a common factor $h^n A_0$, leaves,

$$\omega^2 = \frac{T}{Ma}\left(2 - h - \frac{1}{h}\right). \tag{7.2}$$

This shows that h and $1/h$ give the same normal mode frequency. Conversely, if the frequency ω is fixed, the amplitudes A_n must be an arbitrary linear combination of the amplitudes for h and $1/h$. That is,

$$A_n = \alpha h^n + \beta h^{-n},$$

where α and β are constants.

We will find it convenient to set $h = e^{i\theta}$. The relation giving ω for a given h in equation (7.2) becomes a relation giving ω for a given θ according to,

$$\boxed{\omega^2 = 4\frac{T}{Ma}\sin^2(\theta/2)}. \tag{7.3}$$

The displacement of the nth bead is,

$$u_n = (\alpha e^{in\theta} + \beta e^{-in\theta})e^{i\omega t}. \tag{7.4}$$

7.2.2 Finite System: Boundary Conditions

The value of θ is fixed by the boundary conditions, and this in turn fixes ω. For the string of N beads with both ends fixed, we incorporate the boundary conditions by requiring

$$u_0 = 0, \qquad u_{N+1} = 0.$$

The $u_0 = 0$ condition requires that $\alpha = -\beta$, which makes u_n proportional to $\sin(n\theta)$ only, and the boundary condition at position $N+1$ then imposes,

$$\sin[(N+1)\theta] = 0.$$

This last equation in turn gives

$$\theta = \frac{m\pi}{N+1}, \tag{7.5}$$

where m is an integer which labels the modes.

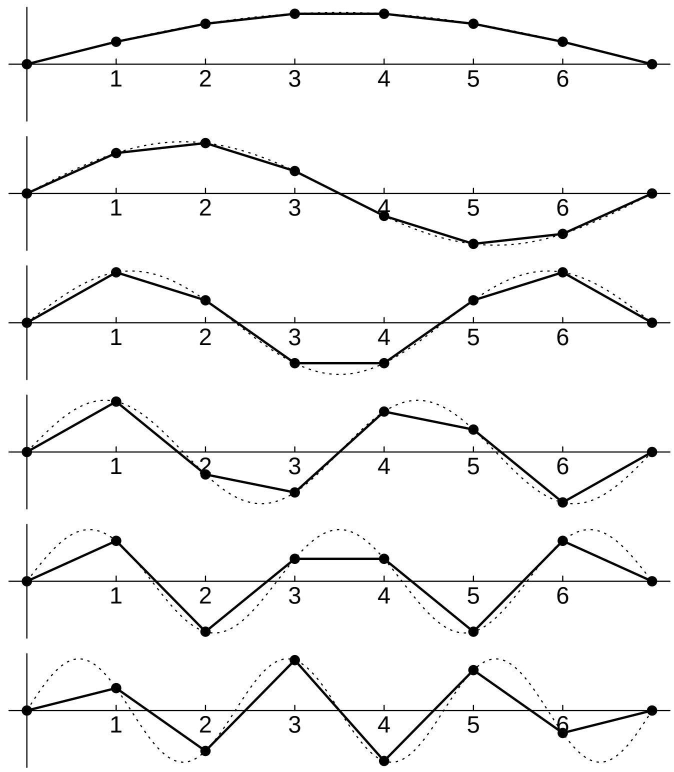


Figure 7.2 The six normal modes of a beaded string fixed at both ends carrying six beads.

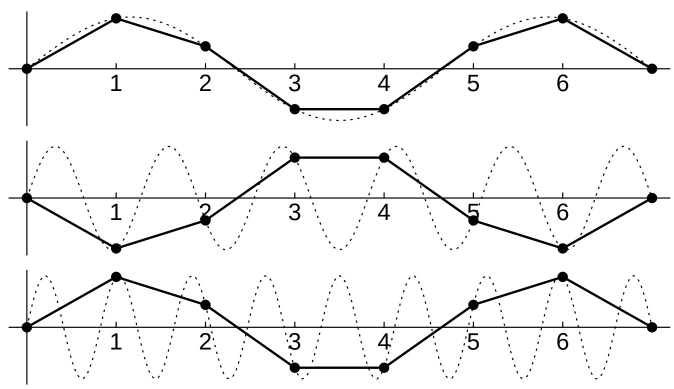


Figure 7.3 Repetition of normal modes for mode numbers greater than six for a string with fixed ends carrying six beads. Modes 3, 11 and 17 are shown. A normal mode remains the same if all the displacements are multiplied by a constant, including -1, so all three modes shown *are* the same.

7.2.3 The Set of Modes

Observe that the linear combination of modes in equation (7.4) is just a sum of left- and right-moving wavelike solutions for the infinite beaded string. For the finite string we are simply constructing a standing wave solution. This is just like finding standing waves for guitar or violin strings or organ pipes, but now the system is discrete rather than continuous.

Look at a string with six beads as an example. There are six degrees of freedom and so we expect six modes as m runs from 1 to 6: these are shown in figure 7.2. The figure also shows the continuous curves obtained by taking n to vary continuously

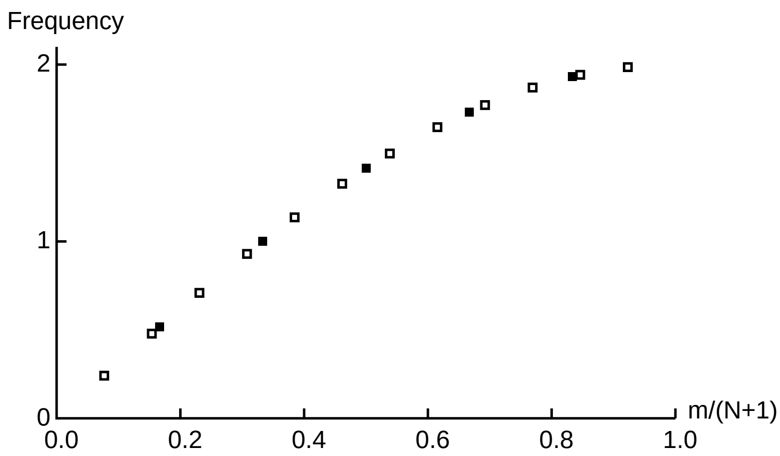


Figure 7.4 Frequencies, in units of $\sqrt{T/Ma}$, of the normal modes of a beaded string with five ($N = 5$, black squares) or twelve ($N = 12$, white squares) beads, showing that the frequencies lie on a universal curve.

and letting na be the position along the string. For larger values of m the modes are repeated (or you get zero displacements). This is shown in figure 7.3. Here you see that the underlying curve of $\sin(n\theta)$ changes, but the positions of the beads, which determine the physical situation are unchanged.

The normal mode frequencies are found by inserting the value of θ from equation (7.5) in equation (7.3) giving ω in terms of θ:

$$\boxed{\omega_m = 2\sqrt{\frac{T}{Ma}}\sin\left(\frac{m\pi}{2(N+1)}\right)}.$$

In figure 7.4 are shown the normal mode frequencies for strings of five ($N = 5$) and twelve ($N = 12$) beads, plotted as functions of $m/(N+1)$. They lie on a universal curve when plotted in terms of this variable. The curve gives the mode frequencies of an infinite line of beads and the finite systems pick out subsets of allowed modes which satisfy the boundary conditions.

A

Supplementary Problems

These are practice questions: you *do not* need to hand in solutions. You can also study past exam papers. PH211 (now PHYS2006) was a new course in 1993, so you'll find some relevant questions in pre 1993 PH101 papers.

1. A rocket burns a kerosene-oxygen mixture: the complete burning of 1 kg of kerosene requires 3.4 kg of oxygen. This burning produces about 4.2×10^7 J of thermal energy. Suppose that *all* of this energy goes into kinetic energy of the reaction products (4.4 kg). What will be the exhaust speed of the reaction products? [Hint: you can answer this by using the expression for the kinetic energy of a system of particles in terms of the centre of mass motion plus that due to the motion relative to the CM.]

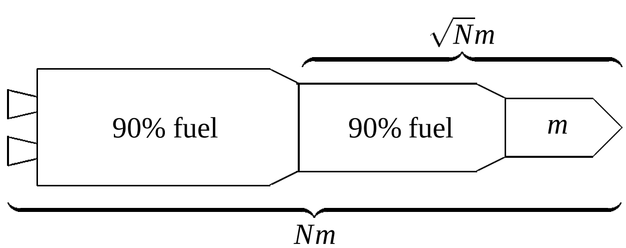
2. At time $t = 0$ a dust particle of mass m_0 starts to fall from rest through a cloud. Its mass grows exponentially with the distance fallen, so that after falling through a distance x its mass is $m_0 \exp(\alpha x)$, where α is a constant. Show that at time t the velocity of the particle is given by

$$v = \sqrt{\frac{g}{\alpha}} \tanh(t\sqrt{\alpha g})$$

where g is the acceleration due to gravity.

3. The total mass of a rocket is 10 kg including fuel. What part of this mass should be fuel in order that the kinetic energy of the rocket after all the fuel is burned is maximised? If the velocity of the exhaust gases is $300\,\mathrm{m\,s^{-1}}$, determine this maximum kinetic energy. Ignore gravity.

4. A payload of mass m is mounted on a two stage rocket. The *total* mass of both rocket stages, fully fuelled, plus the payload, is Nm. The mass of the fully fuelled second stage plus payload is $\sqrt{N}m$. For each stage the exhaust speed is u and the full fuel load makes up 90% of the total mass of the stage.



(i) Show that the speed gained from rest, after first stage burnout and separation followed by second stage burn, is

$$2u\ln\left(\frac{10}{1+9/\sqrt{N}}\right).$$

(ii) If $u = 2.5\,\mathrm{km\,s^{-1}}$, show that the two-stage rocket can achieve a payload velocity of $10\,\mathrm{km\,s^{-1}}$ for large enough N, but that a single stage rocket with the same construction and payload can *never* do so (take the single stage rocket to have payload mass m as before and to have 90% of the stage mass as fuel initially).

5. Find the direction and magnitude of the total torque about the origin produced by the forces $\mathbf{F}_1 = F(\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + \hat{\mathbf{z}})$ acting at $\mathbf{r}_1 = a(\hat{\mathbf{x}} - \hat{\mathbf{y}})$ and $\mathbf{F}_2 = F(2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + 4\hat{\mathbf{z}})$ acting at $\mathbf{r}_2 = a(\hat{\mathbf{x}} + \hat{\mathbf{z}})$.

6. A car travels round a curve of radius r. If h is the height of the centre of mass above the ground and $2b$ the width between the wheels, show that the car will overturn if the speed exceeds $\sqrt{grb/h}$, assuming no side slipping takes place. If the coefficient of friction between the tyres and the road is μ, show that the car will skid before overturning if $\mu < b/h$.

7. (a) A reel of thread of radius a and moment of inertia Mk^2 is allowed to unwind under gravity, the upper end of the thread being fixed. Find the acceleration of the reel and the tension in the thread.

(b) Find the acceleration of a uniform cylinder of radius a rolling down a slope of inclination θ to the horizontal.

8. (a) A mass M is suspended at a distance ℓ from its centre of mass. By writing down the equation of rotational motion, show that the period of small oscillations is

$$2\pi\sqrt{\frac{I}{mg\ell}}$$

where I is the moment of inertia about the point of suspension.

(b) A body of moment of inertia I about its centre of mass is suspended from that point by a wire which produces a torque τ per unit twist. Show that the period of small oscillations is

$$2\pi\sqrt{I/\tau}$$

9. Calculate the moments of inertia of:

(a) a thin rod about its end

(b) a thin circular disc about its axis

(c) a thin circular disc about its diameter

(d) a thin spherical shell about a diameter

(e) a uniform sphere about a diameter

Note that already known results, together with symmetry, may help you.

10. Two cylinders are mounted upon a common axis and a motor can make one rotate with respect to the other. Otherwise the system is isolated. The following sequence of operations takes place:

(a) one half is rotated with respect to the other through angle ϕ

(b) the moments of inertia of the cylinders change from I_1 and I_2 to I_1' and I_2'

(c) the two halves are rotated back until they are in their original *relative* positions

(d) the moments of inertia are restored to their original values.

Show that the whole system is at rest but has rotated through an angle

$$\phi \frac{I_2'/I_1' - I_2/I_1}{(1+I_2/I_1)(1+I_2'/I_1')}$$

This illustrates how a falling cat can manage to land on its feet.

11. A thin straight rod 20 m long, having a linear density λ of $0.5\,\mathrm{kg\,m^{-1}}$ lies along the y-axis with its centre at the origin. A 2 kg uniform sphere lies on the x-axis with its centre of mass 3 m away from the rod's centre of mass. What gravitational force does the rod exert on the sphere?

12. A planet of mass m moves in an elliptical orbit around a sun of mass M. Its maximum and minimum distances from the sun are r_{max} and r_{min}.

Show that the total energy of the planet can be written in the form

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

where $\mathbf{L}$ is the angular momentum. Hence show that

$$r_{\mathrm{max}} + r_{\mathrm{min}} = -\frac{GMm}{E}$$

Using conservation of energy, find the maximum and minimum velocity of the planet (v_{max} and v_{min}).

Assuming Kepler's law relating the period T of the orbit to the semi-major axis of the ellipse, show that

$$T = \frac{\pi(r_{\mathrm{max}} + r_{\mathrm{min}})}{\sqrt{v_{\mathrm{max}} v_{\mathrm{min}}}}$$

13. For motion under a central conservative force, the total energy and the angular momentum $\mathbf{L}$ are conserved. For the special case of an inverse-square law force, such as gravitation or the Coulomb force, with potential energy $V(\mathbf{r}) = -k/r$, we will show that there is a second conserved vector, the Runge-Lenz vector $\mathbf{A}$, given by

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$$

By considering $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \frac{d}{dt}(r^2)$, or otherwise, show that

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{\mathbf{v}}{r} - \frac{\mathbf{r}\cdot\mathbf{v}}{r^3}\mathbf{r}$$

where $\mathbf{v} = \dot{\mathbf{r}}$. Use the equation of motion to show that

$$\dot{\mathbf{p}} \times \mathbf{L} = -\frac{mk}{r^3}\,\mathbf{r} \times (\mathbf{r} \times \mathbf{v})$$

Now use the above results to demonstrate that $\dot{\mathbf{A}} = 0$, or $\mathbf{A}$ is conserved.

$\mathbf{A}$ is perpendicular to $\mathbf{L}$ ($\mathbf{A}\cdot\mathbf{L} = 0$), so $\mathbf{A}$ defines a fixed direction in the orbit plane. Let the angle between $\mathbf{r}$ and $\mathbf{A}$ be θ. Take the dot product of $\mathbf{r}$ with $\mathbf{A}$ to show that

$$rA\cos\theta = L^2 - mkr$$

By comparing to the standard equation, $\ell/r = 1 + e\cos\theta$, express the eccentricity e in terms of the length A of the Runge-Lenz vector. Which point of the orbit is $\mathbf{A}$ directed towards?

Hint: the following identities for arbitrary vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, may be useful

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b}) \\ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

14. **[Hard]** A ballistic rocket is fired from the surface of the Earth with velocity $v < (Rg)^{1/2}$ at an angle α to the vertical. Assuming the equation for its orbit, show that to achieve maximum range, α should be chosen so that $\ell = 2a - R$, where ℓ is the semi-latus rectum, a is the semi-major axis and R is the Earth's radius. Deduce that the maximum range is $2R\theta$ where

$$\sin\theta = \frac{v^2}{2Rg - v^2}$$

15. A locomotive is travelling due North in latitude λ along a straight level track with velocity v. Show that the ratio of the forces on the two rails is approximately

$$1 + \frac{4\omega v h}{ga}\sin\lambda$$

where h is the height of the centre of mass above the rails and $2a$ is the distance between the rails. Calculate this ratio for a speed of $150\,\mathrm{km\,hr^{-1}}$ in latitude 45 deg North, assuming that $h = 2a$. Which rail experiences the larger force?

16. A uniform solid ball has a few turns of light string wound around it. If the end of the string is held steady and the ball allowed to fall under gravity, show that the acceleration of the ball is $5g/7$.

17. A body of moment of inertia I is suspended from a torsion fibre for which the restoring torque per unit angular displacement is T; when the angular velocity of the body is Ω it experiences a retarding torque $k\Omega$. If the top end of the fibre is made to oscillate with angular displacement $\phi_0 \sin\omega t$, where $\omega^2 = T/I$, show that the maximum twist in the fibre is $\phi_0(1 + TI/k^2)^{1/2}$.

18. Two identical masses m are suspended by light strings of length l. The suspension points are distance L apart and a light spring of natural length L and spring constant k connects the two masses. Indicate qualitatively the form of the two normal modes for oscillations in the plane of the strings and spring. Write down the equations of motion for small oscillations of the masses in terms of their horizontal displacements x_1 and x_2 from equilibrium. Find the normal mode frequencies and verify your guess for the ratio x_1/x_2 in the two modes.

19. A large number of identical masses m, arranged in a line at equal intervals a, are joined together by identical springs between neighbours, the springs being such that unit extension requires a force μ. The mass at one end is oscillated along the direction of the line with angular frequency ω. Show that a compressional wave is propagated along the line with wavenumber k given by the expression

$$\omega = \omega_0 \sin(ka/2), \qquad \omega_0 = 2(\mu/m)^{1/2}.$$

What happens if ω is made greater than ω_0?