SEMESTER 1 EXAMINATION 2007/08

CLASSICAL MECHANICS

Duration: 120 MINS

*Answer **all** questions in **Section A** and two **and only two** questions in **Section B**.*

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it. Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

A Sheet of Physical Constants will be provided with this examination paper.

An outline marking scheme is shown in brackets to the right of each question.

Only university approved calculators may be used.

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Section A

A1. The kinetic energy of a system of particles with positions $\mathbf{r}_i$ and masses m_i can be written as

$$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{R}}^2 + \sum_i \frac{1}{2} m_i \dot{\boldsymbol{\rho}}_i^2,$$

where $\mathbf{r}_i = \mathbf{R} + \boldsymbol{\rho}_i$, and $\mathbf{R}$ is the centre of mass coordinate. Give the definition of $\mathbf{R}$ in terms of $\mathbf{r}_i$ and m_i, and a full interpretation of the two terms contributing to T. [4]

A2. State the perpendicular axis theorem for thin flat objects (without proving it). [4]

A3. Explain why spinning skaters rotate faster as they draw their arms in. [4]

A4. Find an expression relating the acceleration $\mathbf{g}$ due to gravity on the Earth's surface to Newton's gravitational constant G and the mass and radius of the earth (assumed spherically symmetric). [4]

A5. What is meant by the term “apparent gravity” as seen by an observer in the rotating frame of the Earth? [4]

Section B

B1. (a) A rocket whose casing and payload have negligible mass, has a mass M of fuel, and is at rest in deep space. It then burns fuel in such a way that the escaping gases have a constant speed u relative to the rocket. Show that when the rocket has mass m the rocket's speed is given by

$$u \ln \left(\frac{M}{m} \right) .$$ [8]

(b) Find the mass of the rocket when its momentum is at a maximum, and thus compute the absolute speed v_e (*i.e.* relative to a stationary observer) of the exhaust gases at this instant. [8]

(c) Suppose now that the exhaust gases are emitted with a relative speed u that *varies* over the flight. Find the absolute velocity v_e of the escaping gases when the momentum is a maximum in this case. [4]

B2. (a) A planet of mass m orbits a sun of mass M subject only to the gravitational attraction of the sun. Explain why the angular momentum $\mathbf{L}$ is conserved, and why this means that the orbit lies in a plane. [4]

(b) Show that the planet's total energy E can be written in the form

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} = E,$$

where G is the universal constant of gravitation and r is the distance from the sun to the planet. [4]

(c) Sketch a graph of the *effective potential*,

$$U(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r},$$

and use your graph to illustrate how different kinds of orbital motion are possible, depending on the value of E. [6]

(d) What minimum total energy, E_{min}, can the motion have for this value of L. What is the shape of the corresponding orbit ? [2]

(e) If $E_{\text{min}} < E < 0$, show that the motion is restricted to have $r_{\text{min}} \leq r \leq r_{\text{max}}$, where,

$$r_{\text{min}} + r_{\text{max}} = -\frac{GMm}{E}.$$

[Ignore any complications due to reduced mass in this question: work in the approximation that the sun remains stationary.] [4]

B3. (a) The equation of motion of a particle moving under gravity with position $\mathbf{x}$ measured from a point on or near the Earth's surface is

$$\ddot{\mathbf{x}} = \mathbf{g}^{*} - 2\,\boldsymbol{\omega} \times \dot{\mathbf{x}}$$

where $\mathbf{g}^{*}$ is the effective local gravitational field and $\boldsymbol{\omega}$ is the angular velocity of the Earth's rotation. Suppose the particle is projected from $\mathbf{x} = \mathbf{a}$ with velocity $\mathbf{v}$. Show that,

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{g}^{*}t - 2\,\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a})$$

in the subseqent motion. If terms of order ω^2 and higher can be neglected, show that,

$$\mathbf{x} = \mathbf{a} + \mathbf{v}t + \frac{1}{2}\mathbf{g}^{*}t^2 - \frac{1}{3}\boldsymbol{\omega} \times \mathbf{g}^{*}t^3 - \boldsymbol{\omega} \times \mathbf{v}\,t^2$$

[10]

(b) A particle is dropped from height h above the earth's surface latitude λ. If all terms of order ω^2 can be neglected, show that the horizontal deflection of the particle when it hits the ground is,

$$\frac{1}{3}\,\omega g \left(\frac{2h}{g}\right)^{3/2} \cos\lambda$$

[8]

(c) In what direction is this deflection ? [2]

[Neglect air resistance in this question.]

B4. (a) Consider the transverse oscillations of a light string of length $6a$ stretched to tension T, carrying five beads, each of mass M, equally spaced distance a apart and distance a from the *fixed* ends of the string. If u_n denotes the transverse displacement of the nth bead, show that the equation of motion for each bead is:

$$\ddot{u}_n = \frac{T}{Ma}(u_{n+1} - 2u_n + u_{n-1}).$$

[4]

(b) What conditions need to be imposed on u_0 and u_6, the displacements of the ends of the string, in order to make the equation of motion correct for the 1st and 5th beads ? [2]

(c) Show that the normal mode frequencies, ω_m, for this system are given by

$$\omega_m^2 = \frac{4T}{Ma}\sin^2\left(\frac{m\pi}{12}\right),$$

where $m = 1, 2, 3, 4, 5.$ [8]

(d) Why is it not necessary to consider other values of m ? [2]

(e) Sketch the displacements of the beads in the five normal modes. [4]

END OF PAPER