Linear Motion of a System of Particles

The linear momentum of a system subject to no net external force is conserved.

$$\boxed{\mathbf{F}^{\text{ext}} = \dot{\mathbf{P}}}$$

$$\boxed{\mathbf{P} = M\dot{\mathbf{R}}} \qquad \boxed{\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}}$$

Centre of Mass

$$\mathbf{R} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i,$$

$$\boxed{\mathbf{P} = M\dot{\mathbf{R}}} \qquad \boxed{\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}}$$

$$\boxed{\mathbf{r}_i = \mathbf{R} + \boldsymbol{\rho}_i}$$

Kinetic Energy

$$\boxed{T = \frac{1}{2}M\dot{\mathbf{R}}^2 + T_{\text{CM}}}$$

Angular Motion of a System of Particles

- Angular equation of motion for each particle i is

$$\mathbf{r}_i \times \mathbf{F}_i = \frac{d}{dt}\left(\mathbf{r}_i \times \mathbf{p}_i\right).$$

- Total angular momentum of the system and total torque are:

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \quad \text{and} \quad \boldsymbol{\tau} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i$$

- Split total force into external and internal parts, same for total torque:

$$\begin{aligned}\boldsymbol{\tau} &= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_i \times \sum_{j \neq i} \mathbf{F}_{ij} \\ &\equiv \boldsymbol{\tau}^{\text{ext}} + \boldsymbol{\tau}^{\text{int}}.\end{aligned}$$

- Use Newton's third law (like linear motion case):

$$\begin{aligned}\boldsymbol{\tau}^{\text{int}} &= \mathbf{r}_1 \times (\mathbf{F}_{12} + \mathbf{F}_{13} + \cdots + \mathbf{F}_{1N}) \\ &\quad + \mathbf{r}_2 \times (\mathbf{F}_{21} + \mathbf{F}_{23} + \cdots + \mathbf{F}_{2N}) + \cdots \\ &= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12} + (\text{other pairs}).\end{aligned}$$

- Assume internal forces act along the lines joining particle pairs, hence $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = 0$: i.e.,

$$\boldsymbol{\tau}^{\text{int}} = 0 \text{ for } \textit{central} \text{ internal forces.}$$

- Obtain:

$$\sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \frac{d}{dt} \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i,$$

  i.e.,

$$\boxed{\boldsymbol{\tau}^{\text{ext}} = \dot{\mathbf{L}}}\,.$$

- Result applies when using coordinates in inertial frame (where Newton's laws apply).

- Used both Newton's third law and the condition that the forces between particles were central.

Angular Motion About the Centre of Mass

- Total angular momentum using the centre of mass (CM) coordinates:

$$\begin{aligned} \mathbf{L} &= \sum_{i=1}^{N} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i = \sum_{i=1}^{N} (\mathbf{R} + \boldsymbol{\rho}_i) \times m_i (\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_i) \\ &= \sum_{i=1}^{N} \mathbf{R} \times m_i \dot{\mathbf{R}} + \sum_{i=1}^{N} \mathbf{R} \times m_i \dot{\boldsymbol{\rho}}_i \\ &+ \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\mathbf{R}} + \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i . \end{aligned}$$

- Second and third terms on the RHS vanish ($\sum m_i \boldsymbol{\rho}_i = 0$ and $\sum m_i \dot{\boldsymbol{\rho}}_i = 0$ from CM definition):

$$\mathbf{L} = \mathbf{R} \times M \dot{\mathbf{R}} + \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i ,$$

write as,

$$\boxed{\mathbf{L} = \mathbf{R} \times M \dot{\mathbf{R}} + \mathbf{L}_{\text{CM}}} .$$

1. First term from CM motion relative to origin (*orbital* angular momentum): *frame dipendent*.

2. Second term from angular motion relative to CM (e.g., spinning planet orbiting around the Sun): *same* in all (inertial) frames and *intrinsic* (or *spin*) angular momentum.

- Take time derivative of the last equation:

$$\begin{aligned}\frac{d\mathbf{L}_{\text{CM}}}{dt} &= \frac{d\mathbf{L}}{dt} - \mathbf{R}\times M\ddot{\mathbf{R}} \quad = \quad \boldsymbol{\tau}^{\text{ext}} - \mathbf{R}\times\mathbf{F}^{\text{ext}} \\ &= \sum_{i=1}^{N}\mathbf{r}_i\times\mathbf{F}_i^{\text{ext}} - \sum_{i=1}^{N}\mathbf{R}\times\mathbf{F}_i^{\text{ext}} \\ &= \sum_{i=1}^{N}(\mathbf{r}_i - \mathbf{R})\times\mathbf{F}_i^{\text{ext}} \\ &= \sum_{i=1}^{N}\boldsymbol{\rho}_i\times\mathbf{F}_i^{\text{ext}} \quad \equiv \quad \boldsymbol{\tau}_{\text{CM}}^{\text{ext}}.\end{aligned}$$

- Find:

$$\boxed{\boldsymbol{\tau}^{\text{ext}} = \dot{\mathbf{L}}} \qquad \text{and} \qquad \boxed{\boldsymbol{\tau}_{\text{CM}}^{\text{ext}} = \dot{\mathbf{L}}_{\text{CM}}}.$$

- Can take moments either about origin of inertial frame or CM.

- *Rationale*:

The angular momentum of a system subject to no external torque is constant.