Rotational Motion of Rigid Bodies

- Rate of change for rotating vector of fixed length:

$$\boxed{\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}}.$$

Moment of Inertia (MoI)

- For a discrete system:

$$\boxed{I \equiv \sum_i m_i R_i^2}.$$

- For a continuous system:

$$\boxed{I = \int_{\text{body}} R^2\, dm = \int_{\text{body}} R^2 \rho\, d^3\mathbf{r}}.$$

Useful definitions

- Radius of gyration k:

$$\boxed{I \equiv Mk^2}.$$

- Angolar momentum along the rotation axis:

$$\boxed{L_n = \sum_i R_i (m_i R_i \omega) = I\omega}.$$

Two Theorems on Moments of Inertia

Parallel Axis Theorem

$$\boxed{I = I_{\text{CM}} + Md^2}.$$

Perpendicular Axis Theorem

$$\boxed{I_z = I_x + I_y}.$$

Example: Spoked Wheel

Angular equation of motion about CM:

$$\tau_{\mathrm{CM}}^{\mathrm{ext}} = I_{\mathrm{CM}}\,\dot{\omega}.$$

Recall MoI of a rod (now of length a),

$$I_{\mathrm{CM}} = Ma^2 + \frac{n}{3}ma^2.$$

Angular equation of motion then gives:

$$Fa = \left(Ma^2 + \frac{n}{3}ma^2\right)\dot{\omega}.$$

Component of linear equation of motion along plane:

$$-F + (M + nm)g\sin\theta = (M + nm)a\dot{\omega}.$$

Eliminate F and solve for $a\dot{\omega}$:

$$a\dot{\omega} = \frac{3(M + nm)g\sin\theta}{6M + 4nm}.$$

Or, conservation of kinetic plus (gravitational) potential energy:

$$\frac{1}{2}(M+nm)v^2+\frac{1}{2}I_{\mathrm{CM}}\omega^2-(M+nm)gx\sin\theta=\text{const},$$

(x distance from starting point).

Use $v=\dot{x}=a\omega$ and differentiate w.r.t. time:

$$\frac{1}{3}(6M+4nm)\dot{x}\ddot{x}=(M+nm)g\sin\theta\,\dot{x},$$

hence

$$a\dot{\omega}=\ddot{x}=\frac{3(M+nm)g\sin\theta}{6M+4nm}.$$

Precession

- Ignore contribution due to slow precession of the top about the vertical axis.

- Torque is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F},$$

where $\mathbf{r}$ is vector from pivot to top's CM and $\mathbf{F} = m\mathbf{g}$ is top's weight.

- In magnitude,

$$\tau = mgr \sin\alpha,$$

where top's axis makes angle α with vertical.

- Top precesses through infinitesimal angle $d\phi$ about vertical axis, then

$$dL = L d\phi \sin\alpha.$$

- If $\dot{\phi} = \omega_{\rm p}$ is precession angular velocity, then

$$\frac{dL}{dt} = L\omega_{\rm p} \sin\alpha.$$

- Apply equation of motion in magnitude:

$$mgr \sin\alpha = L\omega_{\rm p} \sin\alpha.$$

- Precession angular velocity is

$$\boxed{\omega_{\rm p} = \frac{mgr}{L}}.$$

Commentary

1. Final answer independent of α !
2. Steady precession is a special motion: top tends to nod up and down, or nutate, during precession.