Gravitation and Kepler's Laws

- Newton's Law of Universal Gravitation in vectorial form:

$$\boxed{\mathbf{F}_{12} = -\mathbf{F}_{21} = \frac{Gm_1m_2}{r_{12}^2}\hat{\mathbf{r}}_{12}},$$

where the hat (ˆ) denotes a unit vector as usual.

- Gravity obeys the superposition principle, so if particle 1 is attracted by particles 2 and 3, the total force on 1 is $\mathbf{F}_{12} + \mathbf{F}_{13}$.

- Central forces are *conservative*, we can define *gravitational potential energy*:

$$\boxed{V(r) = -\int_{\infty}^{r} \mathbf{F}\cdot d\mathbf{r}' = -\int_{\infty}^{r} (-)\frac{Gm_1m_2}{r'^2}dr' = -\frac{Gm_1m_2}{r}}.$$

- Define also *gravitational potential* (aka gravitational potential energy per unit mass)

$$\boxed{\Phi(r) = -\frac{Gm}{r}}.$$

(set $m_1 = m$ and replace m_2 with 1).

- Likewise, *gravitational field* $\mathbf{g}$ as gravitational force per unit mass:

$$\boxed{\mathbf{g}(\mathbf{r}) = -\frac{Gm}{r^2}\hat{\mathbf{r}}}.$$

- Field and potential are related in the usual way:

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi.$$

Gravity From A Spherical Shell: Direct Calculation

- Consider a thin spherical shell of radius a, mass per unit area ρ and total mass

$$m = 4\pi\rho a^2.$$

- Superposition principle leads to element of mass:

$$dm = \rho\, 2\pi a \sin\theta\, a d\theta = \frac{m}{2}\sin\theta\, d\theta.$$

- Contribution to the potential from annulus is:

$$d\Phi = -\frac{G\,dm}{R} = -\frac{Gm}{2}\frac{\sin\theta\, d\theta}{R}.$$

- Integrate over θ from 0 to π.

- Change the integration variable from θ to R, via

$$R^2 = r^2 + a^2 - 2ar\cos\theta,$$

hence

$$\sin\theta\, d\theta / R = dR/(ar)$$

- If $r \geq a$ integration limits are $r - a$ and $r + a$; if $r \leq a$ they are $a - r$ and $a + r$:

$$\Phi(r) = -\frac{Gm}{2ar}\int_{|r-a|}^{r+a} dR = \begin{cases} -Gm/r & \text{for } r \geq a \\ -Gm/a & \text{for } r < a \end{cases}.$$

- Gravitational field by differentiation:

$$\mathbf{g}(\mathbf{r}) = \begin{cases} -Gm\hat{\mathbf{r}}/r^2 & \text{for } r \geq a \\ 0 & \text{for } r < a \end{cases}.$$

1. Outside the shell, the potential is just that of a point mass at the centre.
2. Inside the shell, the potential is constant and so the force vanishes.

Now Using Analogy With Coulomb Force

- Apply integral form of Gauss' Law to gravitational case:

$$\int_S \mathbf{g}\cdot d\mathbf{S} = -4\pi G \int_V \rho_m \, dV$$

- That is:

Surface integral of normal component of gravitational field over given surface S is equal to $(-4\pi G)$ times the mass contained within surface, with mass obtained by integrating mass density ρ_m over volume V contained by S.

- From spherical symmetry gravitational field $\mathbf{g}$ must be radial:

$$\mathbf{g} = g\,\hat{\mathbf{r}}.$$

- Choose a concentric spherical surface with radius $r > a$: mass enclosed is just shell mass m and Gauss' Law says

$$4\pi r^2 g = -4\pi G m,$$

giving

$$\mathbf{g} = -\frac{Gm}{r^2}\hat{\mathbf{r}} \qquad \text{for } r > a.$$

- Likewise, choose concentric spherical surface inside shell: mass enclosed is zero and $\mathbf{g}$ vanish.

Orbits: Preliminaries

Two-body Problem: Reduced Mass

- Express position $\mathbf{r}_i$ as CM location $\mathbf{R}$ plus displacement $\boldsymbol{\rho}_i$ relative to it:

$$\mathbf{r}_1 = \mathbf{R} + \boldsymbol{\rho}_1, \qquad \mathbf{r}_2 = \mathbf{R} + \boldsymbol{\rho}_2.$$

- Change variables from $\mathbf{r}_1$ and $\mathbf{r}_2$ to $\mathbf{R}$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ (recall $\mathbf{F} = \mathbf{F}_{12} = -\mathbf{F}_{21}$, internal forces only):

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}, \qquad m_2\ddot{\mathbf{r}}_2 = -\mathbf{F}.$$

- Set $M = m_1 + m_2$, thus

$$M\ddot{\mathbf{R}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0,$$

  i.e., CM velocity is constant.

- Consider relative displacement $\mathbf{r}$,

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = (\frac{1}{m_1} + \frac{1}{m_2})\mathbf{F} = \frac{m_1+m_2}{m_1 m_2}\mathbf{F},$$

  i.e.,

$$\boxed{\mathbf{F} = \mu\ddot{\mathbf{r}}},$$

  and re-encounter *reduced mass*

$$\boxed{\mu \equiv \frac{m_1 m_2}{m_1+m_2}}.$$

- For conservative force **F** there is potential energy $V(r)$, hence total energy is:

$$E = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 + V(r).$$

- Likewise, when **F** is central, total angular momentum is

$$\mathbf{L} = M\mathbf{R}\times\dot{\mathbf{R}} + \mu\mathbf{r}\times\dot{\mathbf{r}}.$$

- Since CM velocity is constant, choose inertial frame with origin at **R**, i.e.:

$$\mathbf{R} = 0.$$

- Hence:

$$\boxed{\begin{aligned} E &= \frac{1}{2}\mu\dot{\mathbf{r}}^2 + V(r), \\ \mathbf{L} &= \mu\mathbf{r}\times\dot{\mathbf{r}}. \end{aligned}}$$

- *Rationale*: two-body problem reduces to equivalent single body one of mass μ at position **r** relative to fixed centre, acted upon by force

$$\mathbf{F} = -(\partial V/\partial r)\,\hat{\mathbf{r}}.$$

- If $m_2 \gg m_1$:

$$\mu = m_1 m_2/(m_1 + m_2) \approx m_1,$$

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \approx \mathbf{r}_2$$

(‘fixed Sun and moving Planet approximation’).

- Commentary:

1. Approximation valid for Kepler’s Laws ($m_1 = m_{\text{Planet}}$ and $m_2 = m_{\text{Sun}}$).
2. Can ignore interactions between Planets in comparison to gravitational attraction Planet-Sun.

Two-body Problem: Conserved Quantities

- Gravity is central force: gravitational attraction between two bodies acts along line joining them.

- Gravitational force on mass μ acts in direction $-\mathbf{r}$ and no torque is exerted about fixed centre:

$$\mathbf{L} = \text{ constant.}$$

- Both *magnitude* and *direction* of **L** are fixed !

- Since

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \mu\dot{\mathbf{r}},$$

  L is perpendicular to plane defined by position and momentum of μ.

- Conversely: **r** and **p** must always lie in fixed plane of all directions perpendicular to **L**.

- Can therefore describe motion using plane *polar coordinates* (r, θ), with origin at fixed centre !

- Radial and angular equations of motion become:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{mr^2} \qquad \text{(radial equation)},$$
$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \qquad \text{(angular equation)},$$

- Gravitational force is

$$\mathbf{F} = -k\hat{\mathbf{r}}/r^2, \qquad k = GMm,$$

  wherein $\mu = m = m_{\text{Planet}}$ and $M = m_{\text{Sun}}$.

- Angular equation expresses conservation of angular momentum:

$$L = mr^2\dot{\theta}.$$

- Other conserved quantity is total energy:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - k/r.$$

wherein gravitational potential energy is

$$V(r) = -k/r.$$

- Commentary:

1. Gravitational potential energy will help deducing shape of planetary orbits !

Two-body Problem: Two Problems

Comet

1. Comet approaching Sun in plane of Earth's orbit (assumed circular) crosses orbit at angle of 60° travelling at $50\,\mathrm{km\,s^{-1}}$.
2. Closest approach to Sun is $1/10$ of Earth's orbital radius (r_e).
3. Ignore attraction of comet to Earth compared to Sun (i.e., reduced mass $\mu = m = m_{\mathrm{Comet}}$).
4. *Aim:*

compute comet's speed at point of closest approach.

Solution

- *Key*: angular momentum conservation

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} \quad (\text{of comet about Sun}).$$

- At point of closest approach comet's velocity must be tangential only:

$$|\mathbf{r} \times \mathbf{v}| = r_{\mathrm{min}} v_{\mathrm{max}}.$$

- At crossing point:

$$|\mathbf{r} \times \mathbf{v}| = r_e v \sin 30^\circ.$$

- Equate two expressions:

$$r_{\mathrm{min}} v_{\mathrm{max}} = 0.1\, r_e v_{\mathrm{max}} = \frac{1}{2} r_e v.$$

- Finally,

$$v_{\max} = 5v = 250\,\mathrm{km\,s^{-1}}.$$

Cygnus X1

1. Cygnus X1 is a binary system of a supergiant star of 25 solar masses and a black hole of 10 solar masses, each in a circular orbit about CM with period 5.6 days.
2. Aim:

   Determine distance between supergiant and black hole, given solar mass 1.99×10^{30} kg.

Solution

- *Key*: 2-body equation of motion *in polar coord.s*:

$$\frac{Gm_1m_2}{r^2} = \frac{m_1m_2}{m_1+m_2} r\,\omega^2$$

  ($m_{1,2} \to$ mass, $r \to$ distance, $\omega \to$ ang. velocity).

- <u>Where is RHS 2nd term coming from ?</u>

- Introduce period:

$$T = 2\pi/\omega.$$

- Extract distance:

$$r^3 = \frac{G(m_1+m_2)T^2}{4\pi^2}$$

$$= 27.5 \times 10^{30}\,\text{m}^3.$$

- That is,

$$r = 3 \times 10^{10}\,\text{m}.$$

Kepler's Laws

State Kepler's Laws:

1. The orbits of the planets are ellipses with the Sun at one focus.

2. The radius vector from the Sun to a planet sweeps out equal areas in equal times.

3. The square of the orbital period of a planet is proportional to the cube of the semimajor axis of the planet's orbit ($T^2 \propto a^3$).

Next lecture: their derivation

Kepler's 2nd Law

- This is statement of angular momentum conservation under action of *central* gravitational force.

- Angular equation of motion gives:

$$r^2\dot{\theta} = \frac{L}{m} = \text{const.}$$

- Leads to:

$$\boxed{\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{const}}.$$

Orbit equation

- Ellipses are *specific* to inverse square law for force, hence first and third laws are specific to inverse square law force.

- Study radial equation of motion ($k = GMm$)

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{mr^2} \ !$$

(i) Remove $\dot{\theta}$ using angular momentum conservation,

$$\dot{\theta} = L/mr^2,$$

get

$$\ddot{r} - \frac{L^2}{m^2 r^3} = -\frac{k}{mr^2}.$$

(ii) Use relation

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} = \frac{L}{mr^2}\frac{d}{d\theta},$$

(differential equation for r in terms of θ).

(iii) Substitute $u = 1/r$ to obtain *orbit equation*:

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{mk}{L^2}}.$$

Kepler's 1st Law

- Solution of orbit equation is

$$\frac{1}{r} = \frac{mk}{L^2}(1 + e\cos\theta).$$

- First law: for $0 \leq e < 1$ is an ellipse, with semi latus rectum $l = L^2/mk$.

Kepler's 3rd Law

- Start with 2nd law for rate of area:

$$\frac{dA}{dt} = \frac{L}{2m}.$$

- Integrate over complete orbital period T:

$$T = 2mA/L \ \ (A = \pi ab \text{ is area of ellipse !}).$$

- Substituting for b in terms of a gives third law:

$$\boxed{T^2 = \frac{4\pi^2}{GM}a^3}.$$

Scaling Argument for Kepler's 3rd Law

- Suppose you found a solution to orbit equation
$$\ddot{r} - r\dot{\theta}^2 = -k/mr^2,$$
i.e., r and θ as functions of t.

- Scale radial and variables by constants α and β:
$$r' = \alpha r, \qquad t' = \beta t.$$

- In terms of r' and t', LHS of orbit equation is:
$$\frac{d^2 r'}{dt'^2} - r'\left(\frac{d\theta}{dt'}\right)^2 = \frac{\alpha}{\beta^2}\ddot{r} - \alpha r\left(\frac{\dot{\theta}}{\beta}\right)^2 = \frac{\alpha}{\beta^2}(\ddot{r} - r\dot{\theta}^2).$$

- RHS becomes:
$$-\frac{k}{mr'^2} = \frac{1}{\alpha^2}\left(-\frac{k}{mr^2}\right).$$

- Compare two sides, new solution in terms of r' and t' provided
$$\beta^2 = \alpha^3.$$

- That is,
$$T^2 \propto a^3.$$

1. Need solving orbit equation for proportionality constant.
2. Scaling argument makes clear third law based on inverse-square force law.

Problem Sheet 6

Section B.

- Earth's speed in *circular* motion about Sun !?

- Orbit equation in *polar coordinates*:

$$\frac{l}{r} = 1 \qquad (e = 0)\ !$$

- That is,

$$r_e = l = \frac{L^2}{mk} = \frac{m^2 v_e^2 r_e^2}{m^2 M_{\rm Sun} G}.$$

- Inverting,

$$\frac{v_e^2}{r_e} = \frac{GM_{\rm Sun}}{r_e^2}.$$

- Finally,

$$v_e^2 = \frac{GM_{\rm Sun}}{r_e}.$$

- Everything can be expressed in terms of v_e !

Energy Considerations: Effective Potential

- Gravitational force is conservative, hence total energy E of orbiting body is conserved:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r).$$

- Angular momentum is also conserved (force is central), hence use

$$r^2\dot{\theta} = L/m,$$

to remove $\dot{\theta}^2$:

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r)}.$$

- Formally, energy equation of particle in linear motion under *effective potential*

$$\boxed{U(r) = \frac{L^2}{2mr^2} + V(r)}.$$

- Effective potential contains *centrifugal term*,

$$L^2/2mr^2,$$

arising because *angular momentum is conserved*.

- Replace $V(r) = -k/r$ and use $l = L^2/mk$:

$$U(r) = \frac{kl}{2r^2} - \frac{k}{r} \qquad (\rightarrow \text{ Fig}).$$

Interpret $U(r)$ as a function of r for given E

- By definition $\dot{r}^2 \geq 0$, implying

$$E \geq U(r) = \frac{kl}{2r^2} - \frac{k}{r}.$$

- Draw a horizontal line for E, $U(r)$ lies below it !

Circular Orbit

- At minimum $U(r) = E$, r is constant at

$$r_c = l = L^2/mk,$$

hence orbit is circular and total energy is

$$E = -k/2l = -mk^2/2L^2.$$

Elliptic Orbit

- If $-k/2l < E < 0$, motion is allowed for

$$r_p \leq r \leq r_a,$$

perihelion r_p and aphelion r_a given by roots of

$$E = kl/2r^2 - k/r.$$

Parabolic Orbit

- If $E = 0$, there is always minimum value for r but escape to infinity is just possible.

Hyperbolic Orbit

- For $E > 0$, escape to infinity is possible with finite kinetic energy at infinite separation.

Orbits in a Yukawa Potential

- Consider Yukawa potential

$$V(r) = -\frac{\alpha e^{-\kappa r}}{r} \qquad (\alpha > 0,\ \kappa > 0).$$

- Describes, e.g., attractive force between nucleons in an atomic nuclei.

- Neglect quantum-mechanics and use classical dynamics.

- Effective potential becomes

$$U(r) = \frac{L^2}{2mr^2} - \frac{\alpha e^{-\kappa r}}{r}.$$

- Trajectories are more complicated: $\rightarrow$ Fig.

Interpret $U(r)$ as a function of r for given E

$E < 0$ but greater than U_{min}

- *Rosette* orbit, i.e., ellipse with rotating orientation, aka *precession of perihelion*.
- Typical of small ($\kappa \approx 0$) perturbations of planetary orbits, e.g., due to other planets (irregularities in Uranus' motion led to discovery of Neptune, 1846).

Large r limit

- Term $L^2/2mr^2$ dominates exponentially falling Yukawa term, so $U(r)$ becomes positive !
- If $U_{\text{max}} > E > 0$, two possible orbits classically distinct. In quantum mechanics, 'tunnelling' becomes possible (e.g., alpha decay) !
