Equation of Motion Near Earth's Surface

- Under approximation $x \ll R$:

$$\boxed{m\ddot{\mathbf{x}} = \mathbf{F} + m\mathbf{g}^* - 2m\boldsymbol{\omega} \times \dot{\mathbf{x}}},$$

where we have defined *apparent gravity*,

$$\boxed{\mathbf{g}^* = -g\frac{\mathbf{R}}{R} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})}$$

and *Coriolis force*:

$$-2m\boldsymbol{\omega} \times \dot{\mathbf{x}}.$$

(Boxes from page 39 !)

Apparent Gravity

- $\mathbf{g}^*$ defines local apparent vertical direction ! ($\mathbf{g}$ defines true vertical, towards Earth's centre.)

- $\mathbf{g}^*$ measured by hanging mass from spring so that mass is stationary in rotating frame fixed to Earth:

$$\dot{\mathbf{x}} = 0, \qquad \ddot{\mathbf{x}} = 0.$$

- Magnitude of centrifugal term is,

$$|-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})| = \omega^2 R \cos\lambda.$$

- Direction: see Fig. $\rightarrow$.

- Apply cosine rule to RHS triangle:

$$g^{*2} = g^2 + (\omega^2 R \cos\lambda)^2 - 2g\omega^2 R \cos^2\lambda,$$

hence

$$g^* = g + O(\omega^2).$$

- Apply sine rule to same triangle:

$$\frac{\sin\alpha}{\omega^2 R \cos\lambda} = \frac{\sin\lambda}{g^*}.$$

- Take α small, so $\sin\alpha \approx \alpha$!

- To order ω^2 we can replace g^* by g:

$$\boxed{\alpha = \frac{\omega^2 R}{g} \sin\lambda \cos\lambda}.$$

Commentary

1. Deflection vanishes at equator and poles.
2. Deflection is maximal at latitude 45°.
3. Deflection is governed by

$$\frac{\omega^2 R}{g} = \frac{3.4\,\mathrm{cm\,s^{-2}}}{g} = 0.35\%.$$

(At Southampton, $\lambda = 51^\circ$: $\alpha = 1.7 \times 10^{-3}\,\mathrm{rad}$.)

Coriolis Force

- This is term

$$-2m\boldsymbol{\omega}\times\dot{\mathbf{x}}.$$

- Visualise particle moving diametrically across smooth flat rotating disc with no forces acting horizontally: see Fig. $\rightarrow$.

- Choose convenient set of axes on Earth's surface: see Fig. $\rightarrow$.

- Using this coordinate system, vectors become:

$$\boldsymbol{\omega} = \omega(0, \cos\lambda, \sin\lambda)$$

$$\begin{aligned}\boldsymbol{\omega}\times\dot{\mathbf{x}} &= \omega\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & \cos\lambda & \sin\lambda\\ \dot{x} & \dot{y} & \dot{z}\end{vmatrix}\\ &= \omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda)\hat{\mathbf{x}} + \omega\dot{x}\sin\lambda\hat{\mathbf{y}} - \omega\dot{x}\cos\lambda\hat{\mathbf{z}},\end{aligned}$$

and

$$\mathbf{g}^* = -g^*\hat{\mathbf{z}}.$$

- Equations of motion are:

$$\begin{aligned} m\ddot{x} &= F_x - 2m\omega\left(\dot{z}\cos\lambda - \dot{y}\sin\lambda\right),\\ m\ddot{y} &= F_y - 2m\omega\dot{x}\sin\lambda,\\ m\ddot{z} &= F_z - mg^* + 2m\omega\dot{x}\cos\lambda.\end{aligned}$$

Free Fall — Effects of Coriolis Term

- For particle in *free fall*,

$$\mathbf{F} = 0,$$

equation of motion becomes

$$\ddot{\mathbf{x}} = \mathbf{g}^* - 2\boldsymbol{\omega} \times \dot{\mathbf{x}}.$$

- Work to $O(\omega)$ so that $\mathbf{g}^* \approx \mathbf{g}$.

- Integrate once w.r.t. time with initial conditions

$$\mathbf{x} = \mathbf{a}, \qquad \dot{\mathbf{x}} = \mathbf{v} \qquad \text{at } t = 0$$

(starts with velocity **v** from point **a**).

- Obtain

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{g}t - 2\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a}).$$

- Ignoring terms of $O(\omega^2)$, zero-th order solution

$$\mathbf{x} = \mathbf{a} + \mathbf{v}t + \mathbf{g}t^2/2$$

in cross product term

$$\dot{\mathbf{x}} = \mathbf{v} + \mathbf{g}t - 2\boldsymbol{\omega} \times \left(\mathbf{v}t + \frac{1}{2}\mathbf{g}t^2 \right).$$

- Integrate once more:

$$\boxed{\mathbf{x} = \mathbf{a} + \mathbf{v}t + \frac{1}{2}\mathbf{g}t^2 - \boldsymbol{\omega} \times \left(\mathbf{v}t^2 + \frac{1}{3}\mathbf{g}t^3 \right)}.$$

(Box in page 42 !)

Particle dropped from a tower

- Consider particle dropped from rest from vertical tower of height h:

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}.$$

- Using $\boldsymbol{\omega} \times \mathbf{g} = -\omega g \cos\lambda \hat{\mathbf{x}}$, we find that components x, y and z of $\mathbf{x}$ are:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \frac{1}{2} g t^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3} \omega g t^3 \cos\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

- Particle hits ground when $z = 0$ at $t = \sqrt{2h/g}$.

- For this t, x component of particle's position is

$$\frac{1}{3} \omega \cos\lambda \left(\frac{8h^3}{g} \right)^{1/2}.$$

- Particle strikes ground a little to <u>East</u> of tower's base !

Shell fired from a cannon

- A shell is fired due North with speed v from a cannon, with elevation angle $\pi/4$.

- Take cannon at origin, initial conditions are:

$$\mathbf{v} = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Compute Coriolis' contribution

$$\boldsymbol{\omega} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ 0 & \frac{v}{\sqrt{2}} & \frac{v}{\sqrt{2}} \end{vmatrix}$$

$$= \frac{\omega v}{\sqrt{2}} (\cos\lambda - \sin\lambda)\, \hat{\mathbf{x}}.$$

- Substitute in our solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{vt}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} g t^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$+ \frac{1}{3} \omega g t^3 \cos\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\omega v t^2}{\sqrt{2}} (\cos\lambda - \sin\lambda) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Look at z component:

$$z = vt/\sqrt{2} - gt^2/2 = 0 \quad \longrightarrow \quad t = \sqrt{2}v/g.$$

- Deflection at impact is:

$$x = \frac{\sqrt{2}\omega v^3}{3g^2}(3\sin\lambda - \cos\lambda).$$

- If $3\sin\lambda > \cos\lambda$ then

$$\boxed{\textit{deflection at impact} \text{ is towards East !}}$$

(Happens for $\lambda > \tan^{-1}(1/3) = 18.4^\circ$, λ of Mexico City or Bombay/Mumbai).

- If $3\sin\lambda < \cos\lambda$ (i.e., $\lambda < 18.4^\circ$) then

$$\boxed{\textit{deflection at impact} \text{ is towards West !}}$$

- Deflection along x-direction is sum of positive cubic term plus quadratic term in t:

$$\frac{1}{3}\omega g t^3 \cos\lambda - \frac{\omega v t^2}{\sqrt{2}}(\cos\lambda - \sin\lambda).$$

- Sum is *always* positive for $\lambda > 45^\circ$ (like at So'ton):

$$\boxed{\text{Deflection is Eastward } \textit{throughout trajectory}.}$$

- Now assume $\lambda < 45^\circ$:

$$\boxed{\text{Deflection is } \textit{initially} \text{ to West then } \textit{changes} \text{ to East.}}$$

(See Fig. $\rightarrow$)

Foucault's Pendulum

- See Fig. $\rightarrow$

- Want to describe rotation of oscillation plane.

- Usual choice of coordinates, neglecting $\sim \omega^2$ terms:

$$\hat{\mathbf{z}} = -\mathbf{g}^*/g^* \approx -\mathbf{g}/g.$$

- Consider pendulum of length l free to swing in *any direction.*

- Compute displacement $\mathbf{x}$ of bob using equations of motion in *Cartesian coordinates*:

$$\begin{aligned} m\ddot{x} &= F_x - 2m\omega\left(\dot{z}\cos\lambda - \dot{y}\sin\lambda\right), \\ m\ddot{y} &= F_y - 2m\omega\dot{x}\sin\lambda, \\ m\ddot{z} &= F_z - mg^* + 2m\omega\dot{x}\cos\lambda. \end{aligned}$$

- Assume small oscillations:

$$\boxed{\text{Ignore all } z \text{ terms compared to } x \text{ and } y\text{ !}}$$

- $\mathbf{F}$ is tension of support cable:

$$\begin{aligned} F_x &\approx -mgx/l, \\ F_y &\approx -mgy/l. \end{aligned}$$

- Equation along x and y become:

$$\begin{aligned} \ddot{x} &= -\omega_0^2 x + 2\omega\sin\lambda\dot{y}, \\ \ddot{y} &= -\omega_0^2 y - 2\omega\sin\lambda\dot{x}, \end{aligned}$$

($\omega_0 \equiv \sqrt{g/l}$ pendulum natural angular frequency).

- To solve, define complex quantity $\alpha = x + iy$, hence

$$\ddot{\alpha} + 2i\omega\sin\lambda\dot{\alpha} + \omega_0^2\alpha = 0.$$

- Solution is of type $\alpha = Ae^{ipt}$, provided

$$\begin{aligned} p &= -\omega\sin\lambda \pm \sqrt{\omega_0^2 + \omega^2\sin^2\lambda} \\ &\approx -\omega\sin\lambda \pm \omega_0 \quad (\omega_0 \gg \omega\sin\lambda). \end{aligned}$$

- General solution is

$$\alpha = \left(Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right) e^{-i(\omega\sin\lambda)\,t}$$

  (A and B complex constants).

- Choose initial conditions such that

$$\boxed{\alpha = ae^{-i(\omega\sin\lambda)\,t}\cos(\omega_0 t)}. \quad \text{(Box page 46 !)}$$

Commentary

1. $\cos(\omega_0 t)$ describes usual periodic swing of pendulum.

2. $e^{-i(\omega\sin\lambda)\,t}$ describes rotation of oscillation plane with angular velocity $-\omega\sin\lambda$.