Coupled Oscillators

• Take set of coupled oscillators described by set of generalised coordinates $q_1, \ldots, q_n$.

• In general potential $V(q) \equiv V(q_i,\ i = 1, \ldots, n)$ will be complicated function coupling all oscillators together.

• Consider *small* oscillations about a position of stable equilibrium, e.g., $q_i = 0$ for $i = 1, \ldots, n$.

• Expand potential in Taylor series about this point,

$$V(q) = V(0) + \sum_i \left.\frac{\partial V}{\partial q_i}\right|_0 q_i + \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0 q_i q_j + \cdots.$$

• By adding an overall constant to V we can choose $V(0) = 0$.

• Since we are at position of equilibrium, all first derivative terms vanish.

• Define,

$$K_{ij} \equiv \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0,$$

and drop all remaining terms in expansion.

• Note that K_{ij} is constant symmetric $n \times n$ matrix.

• Corresponding forces are

$$F_i = -\frac{\partial V}{\partial q_i} = -\sum_j K_{ij} q_j$$

and thus equations of motion are

$$M_i \ddot{q}_i = -\sum_j K_{ij} q_j,$$

for $i = 1, \ldots, n$.

• Here M_i's are oscillator masses and K_{ij}'s are 'spring constants'.

• Can put them into matrices !

• For a system of masses connected by springs, with each mass moving in same direction, coordinates can be taken as real positions, then M is diagonal matrix and K is matrix determined by actual spring constants.

Normal modes

- Coupled oscillators !

- Take n of these each with generalised coordinate $q_i(t)$ ($i = 1, \ldots, n$) and define system "position":

$$\mathbf{q}(t) = \big(q_1(t), q_2(t), \ldots, q_n(t)\big).$$

- Differential equations involve time dependence only through time derivatives,

$$\ddot{\mathbf{q}} = -\mathbf{M}^{-1}\mathbf{K}\,\mathbf{q},$$

$$\mathbf{M} = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_n \end{pmatrix}, \mathbf{K} = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{pmatrix}.$$

- Thus can look for time translation invariant solutions, called *normal modes*,

$$\mathbf{q}(t) = \mathbf{A}e^{i\omega t} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} e^{i\omega t},$$

by solving *eigenvalue equation*:

$$\mathbf{M}^{-1}\mathbf{K}\,\mathbf{A} = \omega^2\,\mathbf{A}.$$

- (A_i are *constants*, their *overall* normalisation is arbitrary by linearity of differential equation.)

- *Normal mode*: all oscillators move with *same frequency*, but, in general, *different phases and amplitudes*.

- Normal modes provide all independent solutions of differential equation.

- Once found all normal modes, can construct *any* possible motion as their linear combination !

Example: Masses and Springs

- Take following system: see Fig. $\rightarrow$

- Forces on m_1 and m_2 are

$$F_1 = -k_1 x_1 - k'(x_1 - x_2)$$
$$F_2 = -k_2 x_2 - k'(x_2 - x_1).$$

- Newton's 2nd law in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k' & -k' \\ -k' & k_2 + k' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

- Eigenvalue equation:

$$\begin{pmatrix} (k_1 + k')/m_1 & -k'/m_1 \\ -k'/m_2 & (k_2 + k')/m_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

- Now specialise to case $m_1 = m$, $m_2 = 2m$, $k_1 = k$, $k_2 = 2k$ and $k' = 2k$.

- Eigenvalue equation becomes

$$\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{m}{k} \omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

or, setting $\lambda = m\omega^2/k$,

$$\begin{pmatrix} 3 - \lambda & -2 \\ -1 & 2 - \lambda \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- For there to be a solution, determinant of 2×2 matrix must vanish:

$$\lambda^2 - 5\lambda + 4 = 0,$$

with roots $\lambda = 1$ and $\lambda = 4$.

- Corresponding eigenfrequencies are $\omega = \sqrt{k/m}$ and $\omega = 2\sqrt{k/m}$.

- For each eigenfrequency there is corresponding eigenvector:

$$\omega = \sqrt{\frac{k}{m}}, \qquad \mathbf{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\omega = 2\sqrt{\frac{k}{m}}, \qquad \mathbf{A} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

assuming unit modulus for A_1 and A_2.

1. First normal mode: two masses swing in phase with same amplitude while middle spring remains unstretched.

2. Second normal mode: two masses move out of phase with each other and m_1 has twice amplitude of m_2.

Weak couplings and beats

- Take usual system: see Fig. $\rightarrow$

- Forces on m_1 and m_2 are

$$F_1 = -k_1 x_1 - k'(x_1 - x_2)$$

$$F_2 = -k_2 x_2 - k'(x_2 - x_1).$$

- Newton's 2nd law in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k' & -k' \\ -k' & k_2 + k' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

- Eigenvalue equation:

$$\begin{pmatrix} (k_1 + k')/m_1 & -k'/m_1 \\ -k'/m_2 & (k_2 + k')/m_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

  Now specialise to a case where $m_1 = m_2 = m$, $k_1 = k_2 = k$ and $k' = \varepsilon k$.

- Eigenvalue equation (set $\lambda = m\omega^2/k$):

$$\begin{pmatrix} 1 + \varepsilon - \lambda & -\varepsilon \\ -\varepsilon & 1 + \varepsilon - \lambda \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- From null determinant, quadratic equation for λ:

$$\lambda^2 - 2(1+\varepsilon)\lambda + (1 + 2\varepsilon) = 0,$$

  with roots $\lambda = 1$ and $\lambda = 1 + 2\varepsilon$.

- Hence, eigenfrequencies and normal modes are:

$$\omega_1 = \sqrt{\frac{k}{m}}, \qquad \mathbf{A}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\omega_2 = \sqrt{(1+2\varepsilon)\frac{k}{m}}, \qquad \mathbf{A}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

  1. First mode: two masses swing in phase with same amplitude, central connecting spring remains un-stretched.
  2. Second mode: two masses again have same amplitude, but swing out of phase, alternately approaching and receding from each other.

- For *weak coupling* $\varepsilon \ll 1$, two normal modes have almost same frequency and can observe *beats* if motion contains components from both normal modes.

- Start system from rest by holding left-hand mass at a small displacement d to the right while keeping right-hand mass in equilibrium.

- Let go.

- General solution for motion is:

$$\mathbf{x}(t) = c_1\mathbf{A}_1\cos(\omega_1 t) + c_2\mathbf{A}_2\cos(\omega_2 t) + c_3\mathbf{A}_1\sin(\omega_1 t) + c_4\mathbf{A}_2\sin(\omega_2 t).$$

- System starts from rest: $\frac{d\mathbf{x}(t)}{dt} = 0 \rightarrow c_3 = c_4 = 0$.

- Impose initial conditions:

$$\mathbf{x}(0) = \begin{pmatrix} d \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix} = \frac{c_1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{c_2}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Find solutions: $c_1 = c_2 = d/\sqrt{2}$.

- Motion is given by

$$x_1(t) = \frac{d}{2}\big(\cos(\omega_1 t) + \cos(\omega_2 t)\big),$$
$$x_2(t) = \frac{d}{2}\big(\cos(\omega_1 t) - \cos(\omega_2 t)\big).$$

- Can rewrite as

$$x_1(t) = d\cos\left(\frac{\omega_2 - \omega_1}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right),$$
$$x_2(t) = d\sin\left(\frac{\omega_2 - \omega_1}{2}t\right)\sin\left(\frac{\omega_1 + \omega_2}{2}t\right).$$

- Both x_1 and x_2 has "fast" oscillation at average frequency $(\omega_1 + \omega_2)/2$, modulated by "slow" amplitude variation at frequency $(\omega_2 - \omega_1)/2$ (or vice versa).

- Displacements show contributions of two normal modes beating together (for $\varepsilon = 0.1$, see Fig. $\rightarrow$):

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you better double-check that you have
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