Normal modes of a Beaded String

- Take string stretched to tension T, carrying N beads, each of mass M.

- Beads are equally spaced by distance a and ends of string are at distance a from first/last bead.

- Consider small transverse oscillations of beads with ends held fixed: displacement of nth bead is u_n (see Fig. $\rightarrow$).

- Newton's second law:

$$M\ddot{u}_n = -T\left(\sin\psi + \sin\phi\right).$$

- Limit of small displacements:

$$\sin\psi \approx \frac{u_n - u_{n-1}}{a}, \qquad \text{and} \qquad \sin\phi \approx \frac{u_n - u_{n+1}}{a}.$$

- Equations of motion are:

$$\boxed{\ddot{u}_n = \frac{T}{Ma}(u_{n-1} - 2u_n + u_{n+1})}.$$

- (Note: same equation for longitudinal oscillations of one-dimensional line of masses connected by identical springs: C/M replacing T/Ma – C spring constant of each spring.)

- Boundary conditions are (ends of string fixed):

$$u_0 = 0, \qquad u_{N+1} = 0.$$

- Find normal modes (*i.e, motions where all beads oscillate with same angular frequency* ω):

$$u_n = A_n e^{i\omega t} \qquad (\text{for some } A_n\text{'s}).$$

- Substitute in equation of motion:

$$\omega^2 A_n = \frac{T}{Ma}\left(-A_{n-1} + 2A_n - A_{n+1}\right).$$

- Find A_n's via *recurrence relation* – aka 'discrete form of a differential equation'.

- Boundary conditions become:

$$A_0 = A_{N+1} = 0.$$

- Standard methods:

  1. Treat recurrence relation as matrix equation and determine column vectors of A_n's.
  2. Solve recurrence relations.

- Use *translation invariance* and *local interaction*:

  1. Translation invariance: suppose line of beads be *infinite*, after jumping steps either direction of string system looks identical.
  2. Local interaction: u_n is affected only by u_{n-1}, u_{n+1} (i.e., nearest neighbours) and itself.

- Rationale:

finding combination of normal modes of infinite system satisfying $A_0 = A_{N+1} = 0$ gives normal modes of finite system and don't need to know A_{-1}, A_{N+2} !

- Suppose have found mode for infinite string with displacement amplitudes A_n.

- Shift n-th bead one step left: translation invariance implies system is identical.

- That is, if A_n is of normal mode with frequency ω, shifted A'_n gives another mode with *same* ω:

$$A'_n = A_{n+1}.$$

- Normal mode is arbitrary up to overall scale:

$$A'_n = A_{n+1} = hA_n \qquad (h \text{ complex constant}).$$

- Iterate over n beads:

$$A_n = h^n A_0 \qquad (A_0 \text{ arbitrary}).$$

- Replace A_n's in equation of motion:

$$\omega^2 h^n A_0 = \frac{T}{Ma}\left(-h^{n-1}A_0 + 2h^n A_0 - h^{n+1}A_0\right).$$

- Cancelling common factor $h^n A_0$:

$$\omega^2 = \frac{T}{Ma}\left(2 - h - \frac{1}{h}\right);$$

  hence h and $1/h$ give *same normal mode frequency*.

- Conversely, if ω is fixed, amplitudes A_n must be:

$$A_n = \alpha h^n + \beta h^{-n} \qquad (\alpha \text{ and } \beta \text{ constant}).$$

- Set $h = e^{i\theta}$,

  1. displacement of nth bead is:

  $$u_n = (\alpha e^{in\theta} + \beta e^{-in\theta}) e^{i\omega t};$$

  2. eigenvalue equation becomes:

  $$\boxed{\omega^2 = 4\frac{T}{Ma}\sin^2(\theta/2)}.$$

- Insert boundary conditions

  $$u_0 = 0, \qquad u_{N+1} = 0.$$

  1. First yields:

  $$\alpha = -\beta, \qquad u_n \propto \sin(n\theta).$$

  2. Second imposes:

  $$\sin[(N+1)\theta] = 0 \quad \longrightarrow \quad \theta = \frac{m\pi}{N+1}$$

  (m integer labelling the modes).

- *Normal mode frequencies* are then

  $$\boxed{\omega_m = 2\sqrt{\frac{T}{Ma}}\sin\left(\frac{m\pi}{2(N+1)}\right)}.$$