Quantum Chromodynamics - QCD

Contents

- Evidence for quarks - R, DIS
- Colour Gauge Symmetry
- Gluons
- Asymptotic Freedom
- Inter-quark Forces
- Mesons, and Baryons
- Flavour SU(3)

Learning Outcomes

- Be able to describe the evidence for quarks.
- Be able to write down the QCD covariant derivative and explain the origin of gluons in terms of a gauge symmetry.
- Be able to compute the interaction strength between quarks at tree level in perturbation theory.
- Be able to show how composite quark states break down into different SU(3) colour and flavour representations.
- Be able to describe the possible colour singlet quark states.

Reference Books

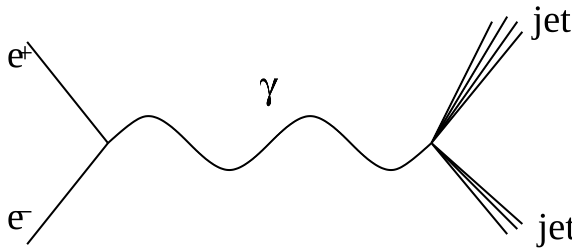
- Introduction to High Energy Physics, Perkins
- Elementary Particle Physics , Kenyon.

QCD

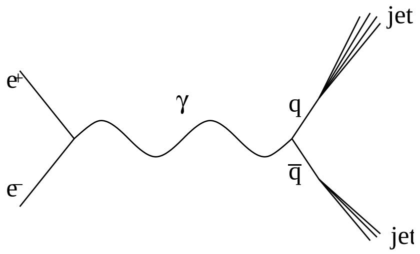
We now turn our attention to the description of the strong nuclear force and hadrons, in terms of quarks and the gauge theory describing their interactions.

1 Evidence For Quarks

Let us begin our study of the strong nuclear force by considering the evidence for the existence of quarks. The simplest studies are through $e^+ \, e^-$ collisions at accelerators where jets of hadrons were discovered



In the 1960s many such hadrons were catalogued including beasts such as $\pi^\pm, \pi^0, p, n, \rho$..... Understanding these jets took a long time but the settled picture is that a quark anti-quark pair are produced in the collision. Their strong interactions then produce more quarks with which they are bound into the hadrons



We will need to build up this picture slowly to understand how quarks are *confined* in hadrons.

1.1 The R-Ratio

If we neglect the jet formation (which is inevitable once the quarks are formed) we can make QED predictions for the jet production cross sections. Remember

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s} \tag{1}$$

so

$$\sigma(e^+e^- \to \bar{q}q) = \frac{4\pi\alpha^2}{3s} \times Q_q^2 \times \text{multiplicity} \tag{2}$$

where the multiplicity is the number of identical copies of a particular quark.

It's useful to define the *R-ratio* as

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \text{multiplicity} \times \sum_f Q_f^2 \tag{3}$$

where Q_f are charges of the quarks which can be produced in the experiment ($Q_u = 2/3, Q_d = -1/3$ etc).

The multiplicity factor surprisingly turns out to be 3 - the first evidence that quarks come in three "colours".

Thus for example in a machine with centre of mass energy $\sqrt{s} = 2GeV$ one can only create u, d, s quarks so

$$R = 3\left[\left(\frac{2}{3}\right)^2 + 2\left(-\frac{1}{3}\right)^2\right] = 2 \tag{4}$$

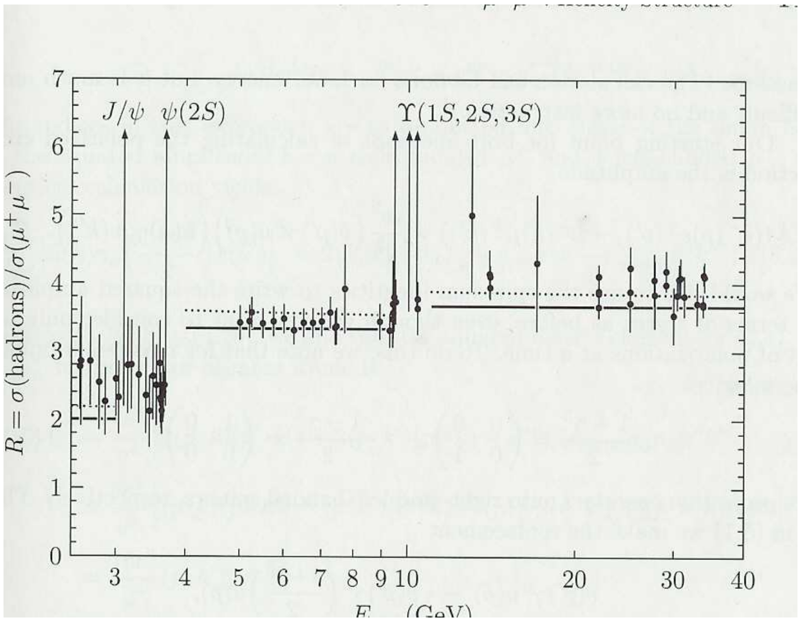
Whereas in a $3GeV$ machine we can also create charm quarks so

$$\Delta R_c = 3\left(\frac{2}{3}\right)^2 = \frac{4}{3} \tag{5}$$

Thus at $3GeV$ we observe

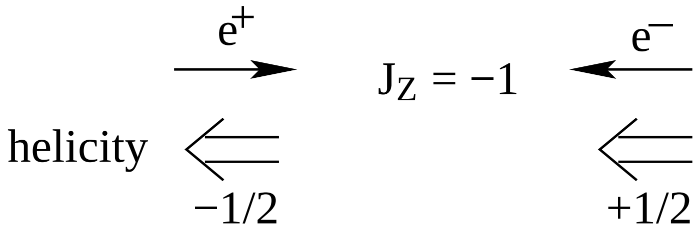
$$R = \frac{10}{3} \tag{6}$$

Here's a plot of real experimental data for R. You can see that above $10GeV$ there is an extra step corresponding to the discovery of the b quark.



1.2 Quark Spin

In e^+e^- collisions the spin 1 nature of the photon created influences the spin structure of the final state particles produced. The preferable configuration for the incoming electron and positron to create the photon is



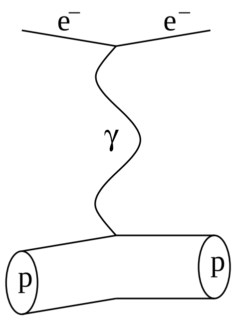
For fermionic final states there is a preference for the particle pair to be produced back to back along the beam line mimicing the electron positron initial state. This shows up in the cross section which goes as

$$\frac{d\sigma}{d\Omega} \sim 1 + \cos^2\theta \tag{7}$$

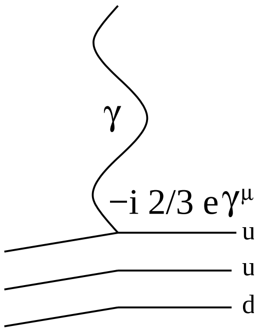
This is the distribution we find for jets which tells us the quarks themselves are spin 1/2 fermions.

1.3 Deep Inelastic Scattering

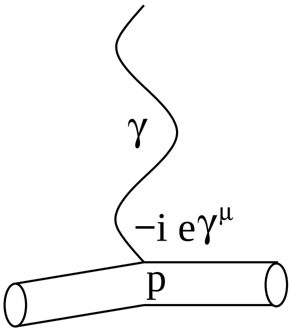
A high energy virtual photon can be used to probe structure within the proton (eg as was done at SLAC in the 1960s)



When a lot of energy is exchanged the photon has a short wavelength ($E = h\nu = hc/\lambda$). The proton appears as a large environment to such photons and the scattering cross section is well described by it hitting an isolated quark (or *parton* as they were initially called)



A very long wavelength photon on the other hand just sees the whole proton with charge $Q_p = 1$.



This transition in the form of the cross-section with scattering energy is observed.

2 Colour Multiplicity

We have seen that there are 3 identical copies (colours) of each quark. We can call them "red", "green" and "blue" but it is important to remember that they are indistinguishable. Any theory we write down must be invariant to interchanging the labels.

In a quantum theory this is even more subtle. We can imagine a state with a wave function

$$\psi = \frac{1}{\sqrt{2}}R + \frac{1}{\sqrt{2}}G \tag{8}$$

which is in a red state 50% of the time and a green state 50% of the time... yet we can't distinguish them so we could equally well describe this as one of our basic quark colour states.

In general there are an infinite set of transformations on the (R,G,B) wave function vector[1] which are described by 3×3 matrices acting on the vector. These matrices are called "elements" of a "group" called SU(3) - these are words borrowed from mathematical group theory but the details of group theory are not essential to us here.

[1]A red quark is represented by the state (1,0,0), a green quark by (0,1,0) etc

These matrices must be unitary as we can see from the following. Both before and after the transformation on our three basis wave functions we require that

$$\int \psi^\dagger \psi \; d^3x = 1 \tag{9}$$

So when we transform $\psi \to \psi' = U\psi$

$$\int \psi'^\dagger \psi' \; d^3x = \int \psi^\dagger U^\dagger U \psi \; d^3x = 1 \tag{10}$$

From which we see that

$$U^\dagger U = 1 \tag{11}$$

It turns out that these infinite set of matrices may be written compactly as

$$U = exp\left[i\theta^a T^a\right] \tag{12}$$

where the index $a = 1..8$ and we are using the index summation convention.

The 8 θ_a are numbers that you can choose any values for - different values give the different U matrices.

The 8 T^a are 3×3 matrices called the "generators" of SU(3). Explicitly they can be written as (normalized so that $TrT^aT^b = \frac{1}{2}\delta^{ab}$)

$$\begin{gathered}
T^1 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\
T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^8 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{gathered} \tag{13}$$

We should also pause to make clear what we mean by the exponential of a matrix in (12) above. We define it as the sum of the infinite series

$$e^M = 1 + M + \frac{M^2}{2} + ... \tag{14}$$

Now the transformation

$$\psi \to \psi' = e^{i\theta^a T^a}\psi \tag{15}$$

looks like a generalization of the gauge transform of QED ($\psi \to e^{iq\alpha(x)}\psi$ - a set of transformations incidentally called the group U(1)). It is therefore a natural step to promote the symmetry to a local level to obtain gauge bosons for the strong nuclear force.

3 QCD

The Dirac equation for the three colour wave functions is

$$(i\partial\!\!\!/ - m)I\psi = 0 \tag{16}$$

where the I is the 3×3 identity matrix in colour space. We will require that this equation is invariant under the transformation

$$\psi \to \psi' = e^{ig_s\theta^a(x)T^a}\psi \tag{17}$$

where g_s will be the strong force's coupling and note the local form of the transformation.

For ease lets just think about a transformation where the θ^a are infinitesimal so

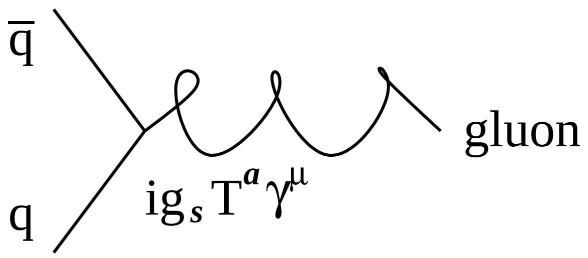
$$\psi \to \psi' = \psi + ig_s\theta^a(x)T^a\psi + ... \tag{18}$$

As usual the ∂^μ operators will not let the $\theta^a(x)$ pass by unhindered... as in QED we must introduce vector fields A^μ into the derivative whose shifts under an enlarged gauge transformation will "undo the damage". Clearly we will now need 8 such vector fields to cancel the 8 degrees of freedom $\theta^a(x)$. Further each θ^a comes with a T^a so so must each of the $A^{\mu a}$ too.

Thus we introduce the covariant derivative

$$\partial^\mu \to D^\mu = \partial^\mu + ig_sG^{\mu a}T^a \tag{19}$$

The 8 $G^{\mu a}$ are the 8 "gluon" fields of QCD. Their Feynman rules to quarks are

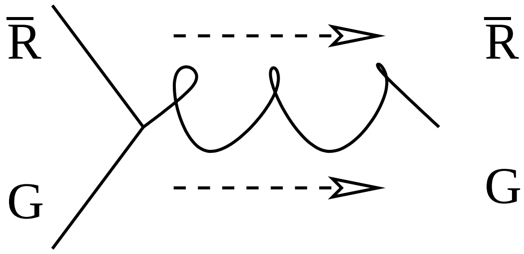


Note that the presence of the generators in the vertices means that the gluons interchange quark colours.

eg $G^{\mu 2}$ has a coupling including T^2 which acts as

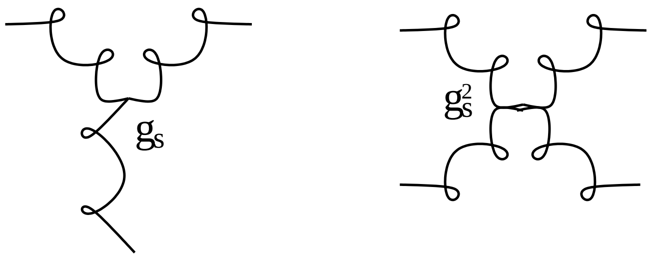
$$T^2\begin{pmatrix} R \\ G \\ B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} R \\ G \\ B \end{pmatrix} = \begin{pmatrix} G \\ R \\ 0 \end{pmatrix} \tag{20}$$

In other words the vertex looks like



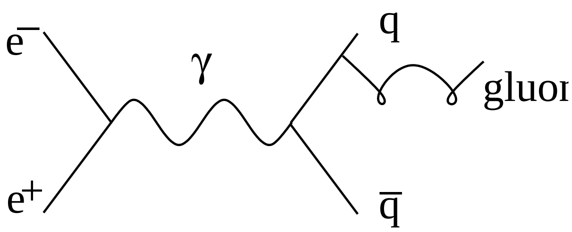
In a sense the gluon here is carrying away G anti-R colour - the gluon is coloured. Note also this guy does not couple to blue quarks.

The mathematics of the generalized Maxwell equations is a quagmire that we shall avoid here. Not surprisingly though having learnt that the gluons are coloured it turns out that they interact directly with each other (unlike photons)



4 Collider Evidence For Gluons

The simplest way to look for gluons is to directly make them. For example



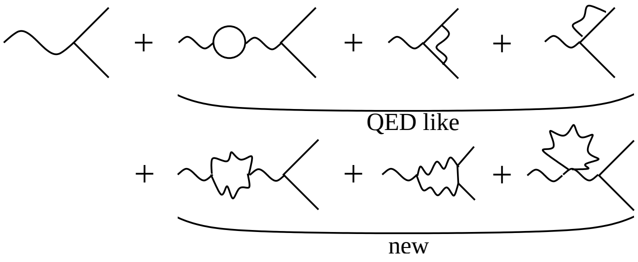
We expect to see three jet events suppressed by g_s^2 (remember to square the diagram to get the probability).

Such events are seen and from them we can extract the strong coupling constant - $\alpha_s(M_Z = 90 GeV) = g_s^2/4\pi \simeq 0.2$.

5 Asymptotic Freedom

The quark and gluon model of QCD is highly successful at predicting results in high energy collider experiments but we need to understand why we never see free quarks or gluons. In fact we never see any colour charged objects in normal life. Formally this is still an open question within QCD but there is enough understanding of the theory to see that it will answer the question. The main concrete understanding (for which the Nobel Prize was awarded in 2004) is asymptotic freedom.

We saw in QED that the running coupling runs with energy scale due to loop corrections. In QCD the same happens but there are additional contributions since the gluons self interact

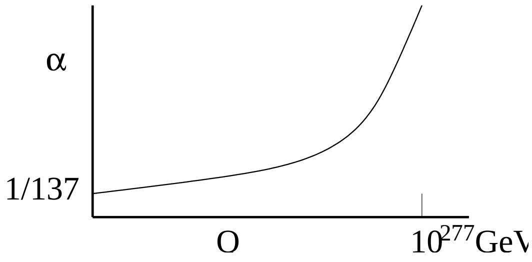


The result of the computation is again that the coupling runs logarithmically

$$\alpha_s(Q^2) = \frac{\alpha_s(M^2)}{1 + \frac{\beta_0}{4\pi}\alpha_s(M^2)\ln Q^2/M^2} \tag{21}$$

where β_0 is the number one extracts from the diagrams above.

In QED just including a single electron $\beta_0 = -4/3$ and the running looks like

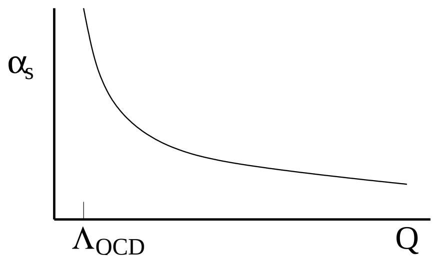


In QCD though the extra contributions enter β_0 with the opposite sign

$$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f \tag{22}$$

where $N_c = 3$, the number of colours, and N_f counts the number of quark flavours accessible at a particular energy scale.

If $N_f < 16$ then $\beta_0 > 0$ and the coupling runs as shown



At large energies or small quark separation the theory is weakly coupled - "asymptotically free".

At low energies or big quark separation the coupling grows. At the scale Λ_{QCD} the equations we have give an infinite value of the coupling. Of course our calculation has been done in perturbation theory which assumes that $g_s \ll 1$ so to extrapolate to this point is beyond the validity of the method... however, that the coupling becomes non-perturbative is known.

Experimental measurement gives $\alpha_s(M_Z) \simeq 0.2$ which implies $\Lambda_{QCD} \sim 200MeV$.

Asymptotic freedom explains much of our picture of QCD. At short distances quarks and gluons are weakly interacting but as they separate the force between them grows. This explains confinement. As we separate a quark and an anti-quark for example the energy of their interaction grows until it will become energetically favourable to pair create a quark anti-quark pair in the middle. The hadron has then broken into two which can then be separated at no extra energy cost. Never is a quark freed.

We can also understand jet formation. The quark anti-quark pair are created close to each other and behave weakly. As they move apart the coupling grows and the energy of that interaction then creates more quark anti-quark pairs and everything pairs up. We see hadronic jets not the original quarks.

5.1 The Role of Λ_{QCD}

We have just seen that QCD "creates" an energy scale - we can trade describing the theory by the value of the dimensionless coupling constant for describing it by the scale Λ_{QCD} at which that coupling blows up. This is called "dimensional transmutation".

The scale Λ_{QCD} created by the running is the only scale in low energy QCD physics. The u, d, s quarks have masses below Λ_{QCD} and can be effectively set to zero - the other more massive quarks we consider not to be part of the low energy theory. We therefore expect that all the masses of QCD bound states will be given by

$$M_{QCD} \sim \#\Lambda_{QCD} \tag{23}$$

where the # is some order one number. Similarly the widths of any resonances will be

$$\Gamma_{QCD} \sim \#\Lambda_{QCD} \tag{24}$$

or in other words the lifetime of an unstable QCD particle will be roughly $1/\Lambda_{QCD}$. This is broadly a correct description of hadronic decay times.

6 Colour Neutrality in QCD

Let's see if we can understand the sorts of hadrons that might favourably form in QCD. Since gluons are so strongly coupled we expect any free colour charges to be attracted to each other very quickly. Nature should only have colour neutral groupings of quarks therefore.

6.1 Neutrality in QED

A neutral state in QED such as an electron and a proton has net charge zero and does not couple to photons. That is to say that under a U(1) QED gauge transformation the wave function of the composite object does not transform. Since it doesn't transform into something else it is a unique state called a "singlet" representation of the transformation. We need to find the equivalent objects for QCD.

6.2 Mesons - Quark Anti-Quark Pairs

What states can we have out of a quark and an anti-quark? The following is a complete list;

$$R\bar{R},\ B\bar{B},\ G\bar{G},\ R\bar{G},\ R\bar{B},\ B\bar{R},\ B\bar{G},\ G\bar{R},\ R\bar{B} \tag{25}$$

In fact when we make colour transformations these states split into two sets. Firstly there is

$$\frac{1}{\sqrt{3}}(R\bar{R} + B\bar{B} + G\bar{G}) \tag{26}$$

This state stays the same and does not transform under the SU(3) rotation and gluons won't couple to it - we call this a "colour neutral" state or "singlet".

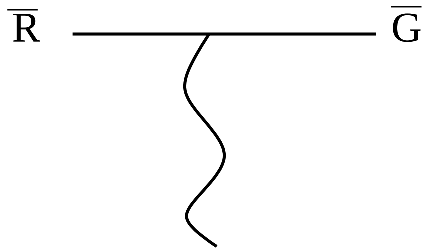
The proof is a little subtle: firstly of course the quark colour vector transforms under a gauge transform as

$$\begin{pmatrix} R \\ G \\ B \end{pmatrix} \to e^{i\theta^a T^a} \begin{pmatrix} R \\ G \\ B \end{pmatrix} \tag{27}$$

The anti-colours actually transform as

$$\begin{pmatrix} \bar{R} \\ \bar{G} \\ \bar{B} \end{pmatrix} \rightarrow e^{-i\theta^a T^{aT}} \begin{pmatrix} \bar{R} \\ \bar{G} \\ \bar{B} \end{pmatrix} \tag{28}$$

The minus sign is because they are anti-particles with opposite charges. The transpose of T^a is the subtle bit - it comes about because when we draw the vertex, for example,



we mean the vertex for a green quark to turn into a red quark but time reversed - the transpose takes this into account.

Now we have

$$(\bar{R}, \bar{G}, \bar{B}) \rightarrow (\bar{R}, \bar{G}, \bar{B}) e^{-i\theta^a T^a} \tag{29}$$

and hence our state

$$(\bar{R}, \bar{G}, \bar{B}) \begin{pmatrix} R \\ G \\ B \end{pmatrix} \rightarrow (\bar{R}, \bar{G}, \bar{B}) e^{-i\theta^a T^a} e^{i\theta^a T^a} \begin{pmatrix} R \\ G \\ B \end{pmatrix} = (\bar{R}, \bar{G}, \bar{B}) \begin{pmatrix} R \\ G \\ B \end{pmatrix} \tag{30}$$

Now, the remaining 8 states in (25) mix into each other though under SU(3) transformations. For example just interchange two labels such as $R \leftrightarrow B$ and you'll see these mix.

$$\frac{1}{\sqrt{2}}(R\bar{R} - B\bar{B}),\ \frac{1}{\sqrt{6}}(R\bar{R} + G\bar{G} - 2B\bar{B}),\ R\bar{G},\ R\bar{B},\ B\bar{R},\ B\bar{G},\ G\bar{R},\ R\bar{B} \tag{31}$$

These states are called an octet of colour and must have identical properties. These appear charged to a gluon.

Formally we write this decomposition as follows

$$3 \otimes \bar{3} = 8 \oplus 1 \tag{32}$$

The presence of the colour neutral singlet matches that we see mesons made of a quark and an anti-quark in nature.

6.3 Baryons - Quark States

There are 9 possible combinations of two quarks too

$$\begin{array}{lll} RR & \frac{1}{\sqrt{2}}(RB+BR) & \frac{1}{\sqrt{2}}(RB-BR) \\ BB & \frac{1}{\sqrt{2}}(RG+GR) & \frac{1}{\sqrt{2}}(RG-GR) \\ GG & \frac{1}{\sqrt{2}}(GB+BG) & \frac{1}{\sqrt{2}}(GB-BG) \end{array} \tag{33}$$

These two groupings are a 6 and a $\bar{3}$. These two "representations" can't mix under SU(3) rotations because one set are symmetric and the other anti-symmetric in the two colours. Under eg $R \leftrightarrow G$ the states within each of the two representations do mix though.

To see that the 3 is an anti-three look at the charge of these states under the diagonal generator T^8. For the 3 (neglecting normalizations)

$$Q_R = +1, \quad Q_G = +1, \quad Q_B = -2 \tag{34}$$

whereas the three states above have charge

$$Q_{RB} = -1, \quad Q_{RG} = +2, \quad Q_{GB} = -1 \tag{35}$$

so they transform as an anti-three.

Since there isn't a colour singlet in $3 \otimes 3$ we do not expect any two quark bound states - it would have colour charge and interact strongly with gluons.

6.4 Three Quark States

We can think of a three quark state as a bound state of a di-quark (the $\bar{3}$ representation) and a further quark. The group theory is one we are familiar with

$$\bar{3} \otimes 3 = 8 \oplus 1 \tag{36}$$

There is a singlet which at the constituent level is given by

$$\begin{aligned} &\frac{1}{\sqrt{3}}\left(R\frac{1}{\sqrt{2}}(GB-BG) + B\frac{1}{\sqrt{2}}(RG-GR) + G\frac{1}{\sqrt{2}}(\ --\ BR\ +\ RB)\right) \\ &= \frac{1}{\sqrt{6}}(RGB - RBG + BRG - BGR\ +\ GRB\ --\ GBR) \end{aligned} \tag{37}$$

Note this is a totally anti-symmetric combination of the colours.

We therefore conclude we expect to find 3 quark states in nature... which we do eg proton - uud, neutron - udd.

The Δ^{++}: There is a state the Δ^{++} with spin $J_z = 3/2$ which is made up of solely up quarks:

$$\mathrm{u}\uparrow \quad \mathrm{u}\uparrow \quad \mathrm{u}\uparrow \tag{38}$$

This provides more evidence for quark colour since this looks like 3 fermions all in the same flavour and spin state! In fact it is totally anti-symmetric since the colour structure is that above - it is an allowed fermionic state.

6.5 Exotics

We can make colour singlets in more exotic fashions eg

Penta-quarks:

We could take four quarks and combine them into two di-quarks in the $\bar{3}$. These di-quarks could then be combined to form a 3. That 3 could then be made into a singlet with an anti-quark!

$$(3 \otimes 3) \otimes (3 \otimes 3) \otimes \bar{3} \;=\; (\bar{3} + ..) \otimes (\bar{3} + ..) \otimes \bar{3} \;=\; (3 + ..) \otimes \bar{3} \;=\; 1 + ... \tag{39}$$

Until recently these states were not seen in the data...

States such as $uuuu\bar{u}$ has the same quantum numbers as the Δ^{++} so would quantum mechanically mix and also decay quickly.

A state such as $uuud\bar{s}$ is more clearly distinct. On the other hand this could fall apart to

$$\theta^{++} \to pK^{+} \tag{40}$$

One might expect this to happen rather fast.

Recent Fuss: a number of experiments in 2005-06 claimed to see the θ^{++} with a very narrow width ($\Gamma < 10MeV$). Other experiments have not seen it since... the verdict is that it does not exist - we expect it to be very unstable to falling apart into a baryon and a meson.

Glueballs

We can also imagine states that are just made from gluons eg $g\bar{g}$ ($8 \otimes \bar{8} = 1 \oplus ...$).

There are some candidates for such states but this picture is naive since such a state has the same quantum numbers as $\bar{u}u$ etc.. presumably the real states are mixtures of all these states.

7 Flavour SU(3)

The SU(3) colour symmetry of QCD resulted from there being three identical copies of the quarks.

"Accidental" symmetries of this sort can also occur. The u,d,s quarks all have masses $\ll \Lambda_{QCD}$. Their QED interactions are much weaker than QCD so as far as the strong force knows these are three identical particles. The upshot is that we expect many different hadrons to have the same masses upto small corrections.

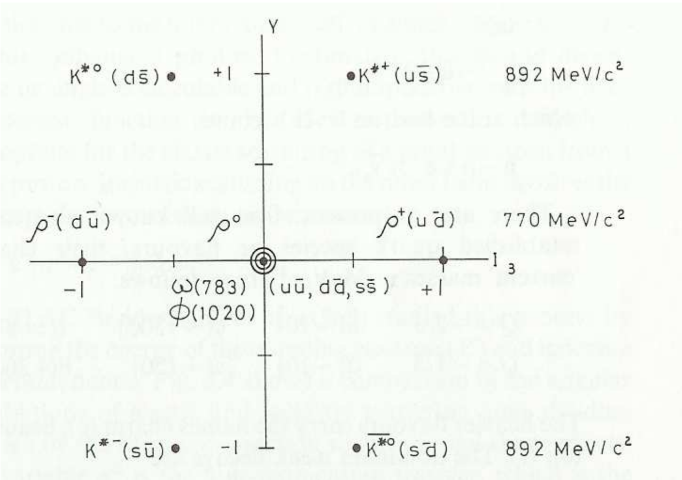
We can keep track of these states using SU(3) flavour symmetries. For example spin 1 quark anti-quark meson states have the flavour structure

$$3 \otimes 3 = 8 \otimes 1 \tag{41}$$

Singlet:

$$\frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s) \rightarrow \phi, \quad 1020 MeV/c^2 \tag{42}$$

Octet :



Here I've labelled the states according to their isospin charge I_3 which is the charge under

$$\frac{1}{2}\begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{43}$$

and hypercharge Y which is the charge under

$$\frac{1}{3}\begin{pmatrix} +1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{44}$$

Note that the octet roughly share the same mass at least in comparison with the splitting of the singlet from them. The whole light hadron spectrum can be classified using SU(3)$_F$.