# Quantum Electrodynamics

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- Review of Classical Electrodynamics
- Quantum Mechanics of the Photon
- Minimal Substitution
- Gauge Invariance
- Scattering Amplitudes in Perturbation Theory
- QED Feynman Rules
- Cross Sections and Decay Rates
- Renormalization and the Running Coupling
- $g - 2$ of the electron

**Learning Outcomes**

- Be able to write down the wave equations of QED and exhibit their gauge invariance.
- Be able to understand the derivation of the scattering amplitude and Feynman Rules.
- Be able to apply the Feynman rules to simple QED diagrams.
- Be able to describe the form of QED cross sections and decay rates
- Be able to explain why the QED coupling runs and how.
- Be able to show that $g = 2$ for the electron at tree level in the Dirac equation

**Reference Books**

- Halzen and Martin

We would now like to include the electromagnetic interactions of particles within the context of the Dirac equation. This will enable us to consider real collider processes such as electron positron annihilation. Firstly though we must think about how to describe photons on their own.

# 1 Photon Wave Equation

To see how to make a relativistic wave equation that describes photons let us begin back at Maxwell's Equations in Differential form

$$\begin{array}{ll} \vec{\nabla}.\vec{E} = \rho & \vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & \vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t} \end{array} \tag{1}$$

I've used units here where $\mu_0 = \epsilon_0 = 1$. We can solve the Maxwell equations with the following potentials

$$\begin{array}{l} \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{array} \tag{2}$$

which are automatically solutions of the Maxwell equations

$$\vec{\nabla}.\vec{B} = \vec{\nabla}.(\vec{\nabla} \times \vec{A}) \equiv 0 \tag{3}$$

and also

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= \vec{\nabla} \times \left(-\tfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\right) \\ &= -\tfrac{\partial(\vec{\nabla} \times \vec{A})}{\partial t} - \vec{\nabla} \times (\vec{\nabla}\phi) \\ &= -\tfrac{\partial \vec{B}}{\partial t} - 0 \end{aligned} \tag{4}$$

This simplifies things greatly since now there are only the remaining two Maxwell equations to solve. Let's write them out in terms of the potentials

$$\vec{\nabla}.\vec{E} = -\nabla^2\phi - \frac{d(\vec{\nabla}.\vec{A})}{dt} = \rho \tag{5}$$

and (since $\vec{\nabla} \times \vec{\nabla} \times \vec{A} \equiv -\nabla^2\vec{A} + \vec{\nabla}.(\vec{\nabla}.\vec{A})$)

$$\vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} = \vec{J} + \frac{\partial}{\partial t}\left(-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\right) \tag{6}$$

or rearranging

$$-\nabla^2\vec{A}+\frac{\partial^2\vec{A}}{\partial t^2}=\vec{J}-\vec{\nabla}\left(\vec{\nabla}.\vec{A}+\frac{\partial\phi}{\partial t}\right) \tag{7}$$

Unfortunately these two equations we are left with are quite messy! To clean them up we can make use of our ability to redefine the potentials whilst keeping the $\vec{E},\vec{B}$ fields the same.

The *gauge transformations* for these potentials that leave $\vec{E},\vec{B}$ invariant are the following

$$\begin{aligned} \vec{A} &\to \vec{A}+\vec{\nabla}\psi \\ \phi &\to \phi-\tfrac{\partial\psi}{\partial t} \end{aligned} \tag{8}$$

Lets chose to make a gauge transformation such that

$$\vec{\nabla}.\vec{A}=-\frac{\partial\phi}{\partial t} \tag{9}$$

In this gauge (Lorentz gauge) Maxwell's equations simplify to

$$-\nabla^2\phi+\frac{\partial^2\phi}{\partial t^2}=\rho \tag{10}$$

$$-\nabla^2\vec{A}+\frac{\partial^2\vec{A}}{\partial t^2}=\vec{J} \tag{11}$$

This form of our remaining Maxwell's equations is much prettier! They also have a very suggestive form for relativity. They suggest we should define

$$J^\mu=(\rho,\vec{J}), \qquad A^\mu=(\phi,\vec{A}) \tag{12}$$

so the Maxwell equations can be written as ($\Box=\partial^\mu\partial_\mu$)

$$\Box A^\mu=J^\mu \tag{13}$$

The $\mu=0$ equation is the $\phi$ equation (10) and the $\mu=1,2,3$ equations give the components of the equation (11) for $\vec{A}$.

The Maxwell equations in Lorentz gauge also required the gauge condition (9) which becomes

$$\partial^\mu A_\mu=0 \tag{14}$$

We will now treat $A^\mu$ as a wave function for photons. In the limit of a large number of photons the wave function can be interpreted as number density. For an observer who is not counting individual photons but just the energy density they provide $A^\mu$ will then look like the classical wave theory. In free space we have

$$\Box A^\mu = 0 \tag{15}$$

with solutions

$$A^\mu = \epsilon^\mu e^{iq.x} \tag{16}$$

where $\epsilon^\mu$ is the polarization tensor and $q^2 = 0$ as required for a photon.

The Lorentz condition enforces

$$q^\mu \epsilon_\mu = 0 \tag{17}$$

and we can choose to set the component of $\epsilon^\mu$ in the direction of motion to zero.

Further within Lorentz gauge there are still gauge transformations

$$A^\mu \to A^\mu + \partial^\mu \chi \quad \text{where} \quad \Box\chi = 0 \tag{18}$$

This can be used to remove one extra degree of freedom from $\epsilon^\mu$ for example in Coulomb gauge setting

$$A^0 = 0 \tag{19}$$

The photon then has two polarizations transverse to the direction of motion.

# 2 Minimal Substitution

We now want to return to thinking about coupling the photons to our Dirac field electrons

$$(i\,\partial\!\!\!/ - m)\psi = 0 \tag{20}$$

The obvious thing to do is to just be led by Lorentz invariance

$$\partial^\mu \to \partial^\mu + ieA^\mu \quad \equiv D^\mu \tag{21}$$

where the factor of $ie$ is a free constant which as our notation suggests will enter as the electric coupling. This is called *minimal substitution* and in fact matches what nature does. We write

$$(i\,D\!\!\!\!/ - m)\psi = 0 \tag{22}$$

We must also include the fermion current in the Maxwell equations which we know are

$$\Box A^\mu = J^\mu \tag{23}$$

We have seen that the probability current for Dirac equation solutions is given by

$$\begin{aligned} \rho &= \psi^\dagger\psi = \psi^\dagger\beta^2\psi = \bar\psi\gamma^0\psi \\ \vec{J} &= \psi^\dagger\vec\alpha\psi = \psi^\dagger\beta^2\vec\alpha\psi = \bar\psi\vec\gamma\psi \end{aligned} \tag{24}$$

When there are many fermions present this becomes the number density current for those particles and so clearly the charge density current should be

$$J^\mu = q\bar{\psi}\gamma^\mu\psi \tag{25}$$

# 3 Gauge Invariance

Minimal substitution in fact works but looks a little ad hoc. It hides a much more fundamental and beautiful symmetry.

Remember that Maxwell's equations are invariant to gauge transformations

$$A^\mu \to A^\mu - \partial^\mu\alpha(x) \tag{26}$$

However, the modified Dirac equation (22) we have written above with $A^\mu$ in it is not invariant to such a transformation. There is though a bigger symmetry which all the equations respect that incorporates the gauge invariance. That larger symmetry is

$$\begin{aligned} \psi &\to e^{iq\alpha(x)}\psi \\ A^\mu &\to A^\mu - \partial^\mu\alpha(x) \end{aligned} \tag{27}$$

**Proof:** We begin with the Dirac equation

$$\left[i\gamma_\mu(\partial^\mu + iqA^\mu) - m\right]\psi = 0 \tag{28}$$

When we make the transformations we arrive at

$$\left[i\gamma_\mu(\partial^\mu + iqA^\mu - iq(\partial^\mu\alpha)) - m\right]e^{iq\alpha(x)}\psi \tag{29}$$

Now for it to be a symmetry we require the solutions of the first equation (28) to also be solutions of the second equation (29).

The way to show this is to try to move the $exp(iq\alpha(x))$ term to the far left. The only term we can not commute it past is the derivative which will act on $\alpha(x)$. In particular

$$\partial^\mu e^{iq\alpha}\psi = e^{iq\alpha}(\partial^\mu\psi) + e^{iq\alpha}iq(\partial^\mu\alpha)\psi \tag{30}$$

However if you look at the term induced in the Dirac equation by the shift in $A^\mu$ you will see it precisely cancels this extra term with $\partial^\mu\alpha$. Thus we arrive at

$$e^{iq\alpha}\left[i\gamma_\mu(\partial^\mu + iqA^\mu) - m\right]\psi = 0 \tag{31}$$

which clearly has the same solutions as this Dirac equation we started with.

The Maxwell equations we already know are invariant to gauge transformations but we must check that $J^\mu = q\bar{\psi}\gamma^\mu\psi$ which we added is too. The exponentials cancel between $\bar{\psi}$ and $\psi$ and all is well.

**The Beauty Herein:** We can look at the gauge transformations from the point of view of the Dirac equation. The free Dirac equation has a symmetry where we shift the solution $\psi$ by a phase $\psi \to e^{i\alpha}\psi$ but where $\alpha$ does not depend on $x$

$$(i\partial\!\!\!/ - m)e^{i\alpha}\psi = e^{i\alpha}(i\partial\!\!\!/ - m)\psi = 0 \tag{32}$$

This is called a *Global* transformation. It is telling us that we are free to place out coordinate axes where we like in the complex plane for $\psi$.

Now in a relativistic theory you might wonder whether two areas of space that are not causally connected should be forced to have the same choice of coordinate axes. You might choose to impose that $\alpha$ can have dependence on spacetime position $x^\mu$. If you tried to impose this you would find it is not a symmetry of the Dirac equation unless you introduced a field $A^\mu$ with the specific transformations we observed nature to have above. In other words you would have had to invent electromagnetism in order have this symmetry. This is what nature does.

Note that in current thinking we view the symmetry as the fundamental guiding theoretical concept of the theory and consider the existence of $A^\mu$ to be derived. We will shortly see other forces which are generalizations of this idea.

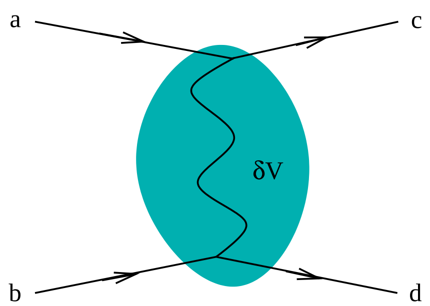
**Massless Photon:** We get one more fact for free too. The Klein Gordon equation for the photon is

$$\Box A^\mu = 0, \qquad \text{not} \qquad (\Box + m^2)A^\mu = 0 \tag{33}$$

The second term would not be gauge invariant so we must set $m^2 = 0$. The symmetry correctly predicts that the photon is massless!

# 4 QED Interactions in Perturbation Theory

The main technique for computations of particle scatterings is perturbation theory - in other words we assume that the coupling $e \ll 1$. We'll be interested in processes such as




Outside the shaded interaction region we assume the particles are free.

Let's write the Dirac equation in a way that displays the smallness of the interaction

$$i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^i\partial_i\psi - m\psi - \gamma^0\ \delta V\ \psi = 0 \tag{34}$$

so for the electromagnetic interaction

$$\delta V = e\gamma^0\gamma^\mu A_\mu \tag{35}$$

Note that $(\gamma^0)^2 = 1$ so the $\gamma^0$ have been included simply for notational convenience.

We will assume that the scattering particles begin in a pure $\vec{p}$ state but the interaction then scatters them to another $\vec{p}$ state with some (small) probability. In general we can write

$$\psi = \sum_n \kappa_n \phi_n(x) e^{-iE_n t} \tag{36}$$

The $\phi_n(x)$ are the free Dirac equation solutions with $n$ labelling the spinor state and the $\vec{p}$ state. The $\kappa_n$ are the probability amplitudes for the given state $n$. Before the interaction all the $\kappa_n$ will be zero except one but during the interaction ($-T/2 < t < T/2$) we allow $\kappa_n$ to change - $\kappa_n(t)$.

If we now substitute the solution into the perturbed Dirac equation above then, at leading order, we obtain zero since we have expanded in solutions of the unperturbed equation. At next order we find

$$i\gamma_0 \sum_n \left(\frac{d\kappa_n}{dt}\right) \phi_n e^{-iE_n t} = \sum_n \gamma_0\ \delta V\ \kappa_n \phi_n(x) e^{-iE_n t} \tag{37}$$

Now we will make use of the orthogonality of the $\phi_n$ to extract the final state $\kappa_n$. We multiply through by $(-i)\ \int d^3x\ \phi_f^\dagger \gamma_0$

$$\frac{d\kappa_f}{dt} = -i\sum_n \kappa_n \int d^3x\ \phi_f^\dagger \delta V \phi_n\ e^{-i(E_n - E_f)t} \tag{38}$$

For a discussion of normalization of the spinors see section 2.5 (we're using $N = 1/\sqrt{2EV}$).

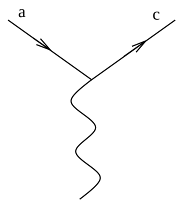
Remembering that at $t = -T/2$ $\kappa_i = 1$ and $\kappa_{i\neq n} = 0$ at leading order we have (upon integrating over $\int_{\frac{-T}{2}}^{\frac{T}{2}} dt$)

$$\frac{d\kappa_f}{dt} = -i\int\ \psi_f^\dagger \delta V \psi_i\ d^3x \tag{39}$$

and integrating with respect to $t$ we find the important result

$$\kappa_f(T/2) = -i\int\ \psi_f^\dagger \delta V \psi_i\ d^4x \tag{40}$$

Now let's use our explicit form for $\delta V$ in QED and concentrate on the scattering of a particle $a \to c$ by a photon $A^\mu$



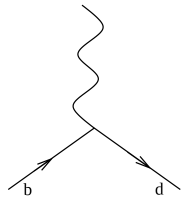

$$\begin{aligned}\kappa_{ca} &= -i \int \bar{\psi}^c (e\gamma_\mu A^\mu) \psi^a \; d^4x \\ &= -i \int J_\mu^{ca} A^\mu \; d^4x\end{aligned} \tag{41}$$

where

$$J_\mu^{ca} = e \; \bar{\psi}^c \gamma_\mu \psi^a = e \; N_a N_c \; \bar{u}^c \gamma_\mu u^a \; e^{i(p_c - p_a).x} \tag{42}$$

The $N$s here are the normalizations of the spatial wave functions $\psi$ again from section 2.5.

We're really interested in two particles scattering off each other so we'd better compute the $A^\mu$ field produced when another particle scatters from state $b \to d$




$$\Box A^\mu = J_{db}^\mu = e \; N_b N_d \; \bar{u}_d \gamma_\mu u_b \; e^{i(p_d - p_b).x} \tag{43}$$

the solution is

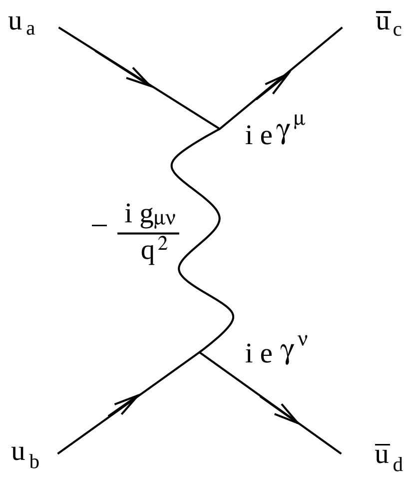
$$A^\mu = -\frac{1}{q^2} J_{db}^\mu, \qquad q = p_d - p_b \tag{44}$$

So finally substituting this back into our expression for $\kappa_{ca}$ we find

$$\kappa_{fi} = -i \; N_a N_b N_c N_d \; \bar{u}^c (e\gamma_\mu) u^a \; \left( -\frac{1}{q^2} \right) \; \bar{u}^d (e\gamma^\mu) u^b \; \int e^{i(p_c + p_d - p_a - p_b).x} d^4x \tag{45}$$

Note that the integral is just a delta function that ensures 4-momentum conservation in the interaction.

In order to make this result more memorable Feynman developed his famous rules that associate different parts of the expression with elements of a diagram of the scattering.



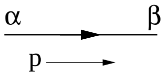
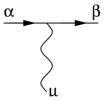

where momentum is conserved at the vertices.

Multiplying these rules out gives us $-i\mathcal{M}_{fi}$ where

$$\kappa_{fi} = -i\ N_a N_b N_c N_d\ (2\pi)^4 \delta^4(p_f - p_i)\ \mathcal{M}_{fi} \tag{46}$$

## 4.1 Summary of Feynman Rules of QED

The Feynman rules for computing the amplitude $\mathcal{M}_{fi}$ for an arbitrary process in QED are summarized in Table 1. I have also included the rules for internal fermion lines and external photons though we will not derive them in this course.

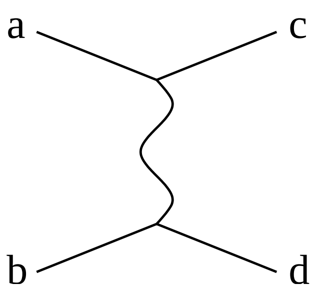
| For every ... | draw ... | write ... |
|---|---|---|
| Internal photon line |   | $\dfrac{-ig^{\mu\nu}}{p^2 + i0^+}$ |
| Internal fermion line |   | $\dfrac{i(\not p + m)_{\alpha\beta}}{p^2 - m^2 + i0^+}$ |
| Vertex |   | $-ie\gamma^{\mu}_{\alpha\beta}$ |
| Outgoing electron | | $\bar{u}_\alpha(s,p)$ |
| Incoming electron | | $u_\alpha(s,p)$ |
| Outgoing positron | | $v_\alpha(s,p)$ |
| Incoming positron | | $\bar{v}_\alpha(s,p)$ |
| Outgoing photon | | $\varepsilon^{*\mu}(\lambda,p)$ |
| Incoming photon | | $\varepsilon^{\mu}(\lambda,p)$ |

- Attach a directed momentum to every internal line
- Conserve momentum at every vertex, i.e. include $\delta^{(4)}(\sum p_i)$
- Integrate over all internal momenta

Table 1: Feynman rules for QED. $\mu$, $\nu$ are Lorentz indices, $\alpha$, $\beta$ are spinor indices and $s$ and $\lambda$ fix the polarization of the electron and photon respectively.

# 5 Cross Sections and Decay Widths

We saw that the probability for a transition in a two particle to two particle scattering was give by




$$-i\mathcal{M} = \bar{u}_c(iQ\gamma^\mu)u_a \left(-\frac{i}{q^2}\right) \bar{u}_d(iQ\gamma_\mu)u_b \tag{47}$$

Given the particular external state solutions of the Dirac equation we can just calculate this number. This is done in detail in the books. The answers are actually quite compact. For example one finds for the process $e^-\mu^- \to e^-\mu^-$ (in the limit where we neglect the masses)

$$|\mathcal{M}|^2 = 2e^4 \frac{s^2 + u^2}{t^2} \tag{48}$$

The result is expressed in terms of

**Mandelstam Variables**:

$$s = (p_a + p_b)^2 \tag{49}$$

$$t = (p_a - p_c)^2 \tag{50}$$

$$u = (p_a - p_d)^2 \tag{51}$$

Note

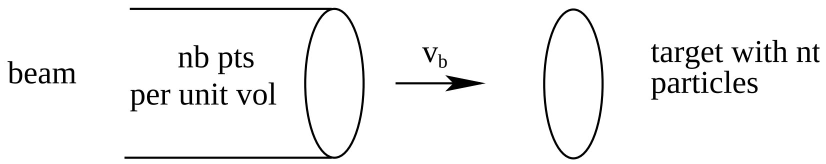
$$\begin{aligned} s + t + u &= 3p_a^2 + p_b^2 + p_c^2 + p_d^2 + 2p_a.p_d - 2p_a.p_c - 2p_a.p_d \\ &= m_a^2 + m_b^2 + m_c^2 + m_d^2 + 2p_a^2 + 2p_a.(p_b - p_c - p_d) \\ &= m_a^2 + m_b^2 + m_c^2 + m_d^2 \end{aligned} \tag{52}$$

In fact the result above for $\mathcal{M}$ is *spin averaged*. This means it is the result for an experiment in which the initial spin state is random and the final spin state is not measured. One therefore averages the result for the initial spin and sums over the final spin states.

In practice one does not scatter a single particle off another single particle since this is hard to arrange and since most of the time nothing would happen. Instead bunches of particles are collided with each other or a static target. The actual measurement made is of:

**Cross-Sections**

A typical experiment can be schematically represented as



We have

$$\begin{array}{ccccccc} \#\text{ scatters/sec} & = & \text{flux} & \times & \#\text{ target pts} & \times & \text{cross} - \text{section}, \sigma \\ [T]^{-1} & & [L^2T]^{-1} & & & & [L]^2 \end{array} \tag{53}$$

where I have displayed the dimensions of the terms. Algebraically we could write

$$n_s = n_b v \; n_t \; \sigma \tag{54}$$

Note that in a classical scattering of say balls there is a scattering with probability one if one ball hits any of the area of another. In this case the cross section is precisely the area the ball presents to the scatterer - hence the name.

Experimenters measure

$$\sigma = \frac{\#\text{ scatters per sec}}{\text{flux} \times \#\text{ target pts}} \tag{55}$$

The number of scatters/sec depends on our probability $|\mathcal{M}|^2$ summed over all possible final states. Frequently the number of scatters into some solid angle is measured so people quote the differential cross-section $\frac{d\sigma}{d\Omega}$. Again we will not explicitly calculate these factors in this course but they are not particularly subtle, just laborious.

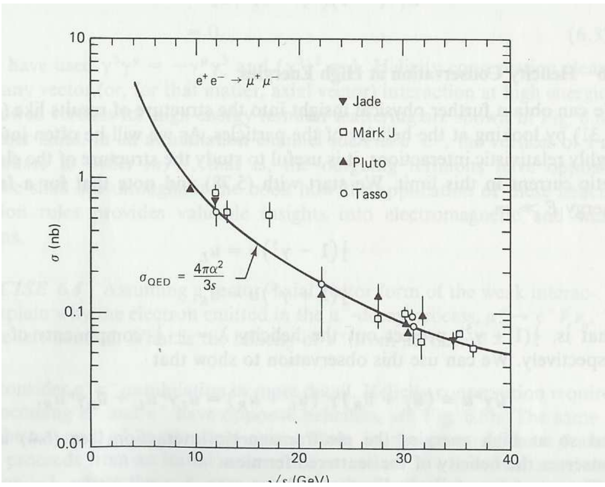
So for example consider the process $e^+e^- \to \mu^+\mu^-$

$$|\mathcal{M}|^2 = 2e^4 \frac{t^2 + u^2}{s^2} \tag{56}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} 2e^4 \left[\frac{1}{2}(1 + \cos^2\theta)\right] \tag{57}$$

$$\sigma = \frac{4\pi\alpha^2}{3s} \tag{58}$$

Below I plot data for this process from the experiment Petra - the theory fits the data well.



**Decay Rates**

When observing particles decays

$$A \to 1 + 2 + ... \tag{59}$$

one measures the number of decays per second per number of $A$ in the sample. This is again just the probability $|\mathcal{M}|^2$ summed over all possible final states. So we measure

$$\Gamma = -\frac{dN_A}{dt}/N_A \tag{60}$$

Integrating gives

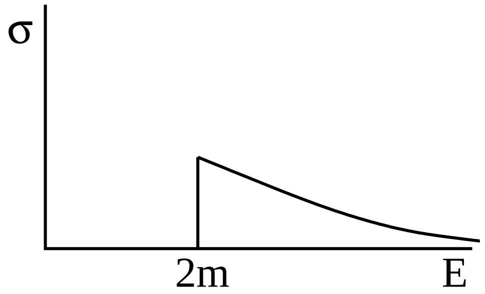
$$N_A(t) = N_A(0)e^{-\Gamma t} \tag{61}$$

So $\Gamma^{-1}$ is the life-time of the particle since $1/e$ of the particles decay in that time.

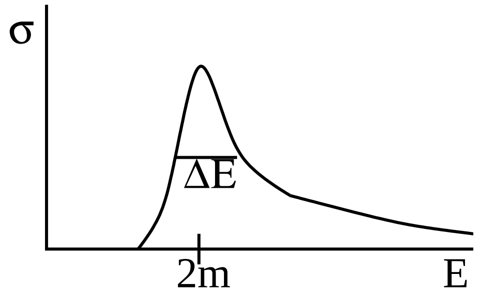
$\Gamma$ is also referred to as the *width* of the decaying particle. To see why, imagine searching for the particle $A$ through

$$e^+e^- \to A \to \text{decay products} \tag{62}$$

You might expect to find a cross-section vs energy that is zero until you have enough energy to create $A$, then a sharp edge at $E = 2m_A$.




In fact rather than this edge one finds a peak.



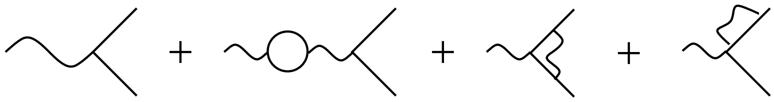

Close to $E = 2m_A$ the cross section is much higher than you might naively expect due to a resonance effect. The width of the resonance peak is determined from the Uncertainty Principle using $\Gamma^{-1}$ as the uncertainty in time

$$\Delta E \Delta t \sim \hbar \tag{63}$$

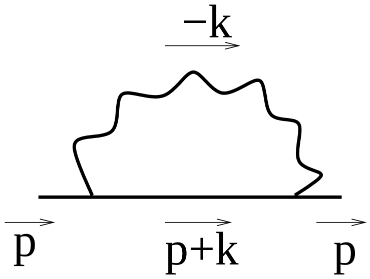
The width of a resonance therefore counts the number of decay channels a particle has!

# 6 Renormalization

When we want to calculate scattering amplitudes beyond $\mathcal{O}(e^2)$ we encounter loop diagrams



Such a loop has a free momentum in it




Quantum mechanically we should allow all possible states in the loop.. but since there are potentially an infinite number of possible momenta the answer after doing the sum is infinity! The diagram we have drawn contributes to a freely travelling electron and conspires to make the mass infinite and the normalization of the wave function infinite. What's going on?

Actually this is an example of a problem we have in classical physics too. If we treat the electron as a charged ball it has some energy

$$E_{\text{sphere}} = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R} = mc^2 \tag{64}$$

If we believe the electron is a point like particle we find it has an infinite mass.

All we are learning in these examples is that we are totally ignorant (in both EM and QED) of high energy (ultra-violet) physics that really determines the electron mass. Indeed no one would suggest that QED is a good theory at any energy scale - at the weak scale we must include the weak force and at very high energies gravity.

What we do in both cases is then to "ignore" this contribution we can't compute. Formally we can write everywhere in the equations

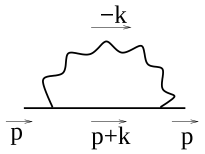
$$m_{\text{physical}} = m_{bare} + e^2 \log \infty \tag{65}$$

where the "bare" mass is the one we'd put into the Dirac equation before we did this computation. It's important that everywhere the physical mass appears there is the same infinite expression but if that happens we can just call the whole lot the observed mass.

If this is going to work we'd better be able to absorb all divergences into the four parameters of the theory - the electric charge and mass of the electron and the wave function normalizations of the electron and photon. In fact it is a tougher task to ask of the theory because this must continue to be true at higher loop level where new divergences appear. Remarkably this has been shown to be the case - gauge theories such as QED are unique in having this property as far as we know.

This looks miraculous but in fact physically it is telling us that we have the sensible infra-red end of a correct theory. We call this "bit" of a theory valid only at low energies an "effective" theory. We can use this theory in the infra-red and all our ignorance of the ultra-violet is hidden in the parameters of the theory that are handed to us by nature.

All of this though does not mean we just drop the loops in our computations. When the momenta in the loop are of low energy the theory is the correct description. So we might calculate



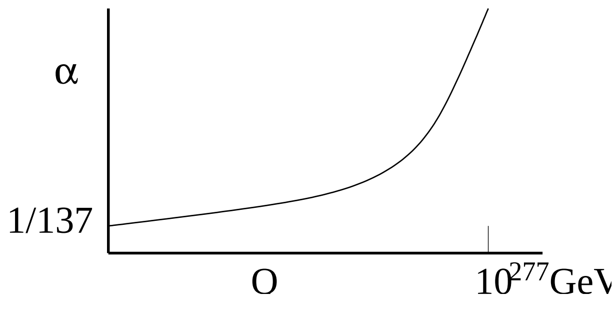

$$\sim e^2 \log \frac{\Lambda^2}{p^2} \tag{66}$$

and shift the mass, say, by this amount. Whilst, when the external particle has momentum $q$ we would get a $q$ rather than $p$ in the answer and want to shift the mass by a different amount. The difference between these two amounts we want to shift by is real and tells us that the particle has a different mass at different energies. The couplings in QED are energy scale dependent.

For example in QED the coupling constant "runs" with energy according to

$$\alpha(Q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \ln \frac{Q^2}{M^2}} \tag{67}$$

where $\alpha_0$ is the value of the coupling at the sale $M$. Plotting the solution gives




Note there is a "Landau pole" at very high energies where the coupling blows up. This is a ridiculously high scale though of $10^{277}$GeV. QED will surely be replaced by something else before then!

# 7 $g-2$ of the Electron

The classic success of renormalization in QED is in calculating the gyromagnetic ratio of the electron. Lets sketch how this story goes. The interaction amplitude between an electron and a photon is given by (41)

$$\kappa_{fi} = -i \int J^{fi\ \mu} A_\mu d^4x \tag{68}$$

where

$$J^{fi\ \mu} = -e\bar{\psi}_f \gamma^\mu \psi_i = -e\bar{u}_f \gamma^\mu u_i e^{i(p_f - p_i).x} \tag{69}$$

There are actually two sorts of interaction present here as we can see using the Gordon decomposition

$$\bar{u}_f \gamma^\mu u_i = \frac{1}{2m} \bar{u}_F \left[ (p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu \right] u_i \tag{70}$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{71}$$

The Gordon decomposition can be easily derived using the Clifford algebra and the fact that $\not{p}_i u_i = m u_i$.

Inserting the Gordon decomposition into our interaction expression gives us two types of term. The first takes the form

$$J^{fi\ \mu} A_\mu \;\; = \;\; -\, e\bar{u}_f \frac{1}{2m} (p_f + p_i)^\mu u_i e^{i(p_f - p_i).x} A_\mu \tag{72}$$

which is diagonal in spinor space. This is just the electric coupling of a Klein Gordon type field.

The second term involves the spin structure and is therefore unique to fermions. We have

$$\kappa_{fi} = -i2\pi\delta(E_i - E_f) \int J^{fi\mu} A_\mu d^3x \tag{73}$$

where the time integration has been explicitly carried out to give the energy conserving $\delta$ function, and the spatial integral is

$$\int J^{fi\mu} A_\mu d^3x \;\; = \;\; -\frac{e}{2m} \int \bar{\psi}_f i\sigma^{\mu\nu} (p_f - p_i)^\nu \psi_i A^\mu d^3x \tag{74}$$

To understand this term better we must take the non-relativistic limit. A number of simplifications result

- the delta function sets $E_f = E_i$ so $(p_f - p_i)^0 = 0$
- the spinors are close to static solutions $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ so we can drop the bottom two components

- let's also look at the coupling to a time independent magnetic field so $A^\mu = (0, \vec{A})$

These restrictions mean that $\mu$ and $\nu$ must be spatial indices. Using the explicit form of the $\gamma$ matrices and restricting to just the top right $2 \times 2$ matrix that acts on the top two components of the spinor we have

$$\gamma^0[\gamma^i, \gamma^j] = \begin{pmatrix} -[\sigma^i, \sigma^j] & \dots \\ \dots & \dots \end{pmatrix} \tag{75}$$

The upshot of this index structure is that

$$\kappa_{fi} = -i2\pi\delta(E_i - E_f) \int \psi_f^\dagger \left( \frac{e}{2m} \vec{\sigma}.(\vec{\nabla} \times \vec{A}) \right) \psi_i d^3x \tag{76}$$

where $\psi$ now has only two components. This is a coupling to the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. It takes the form of a magnetic moment interaction

$$-\vec{\mu}.\vec{B} \tag{77}$$

and we see that we're predicting

$$\vec{\mu} = -\frac{e\hbar}{2mc}\vec{\sigma} \tag{78}$$

In classical physics the magnetic moment of an orbiting charge $e$ is usually written

$$\vec{\mu}_{\text{orb}} = -\frac{e}{2mc}\vec{L} \tag{79}$$

and by analogy experimentalists defined the magnetic moment due to intrinsic spin of the charge as

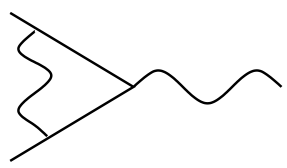
$$\vec{\mu}_{\text{spin}} = -\frac{ge}{2mc}\vec{S} = -\frac{ge}{2mc}\frac{\vec{\sigma}}{2} \tag{80}$$

where $g$ is the gyromagnetic ratio of the particle. The Dirac equation predicts

$$g_{\text{Dirac}} = 2 \tag{81}$$

Experimentally one finds for the electron that $g - 2 = 0.00232$ which is pretty good already.

The discrepancy though is due to the next order diagram



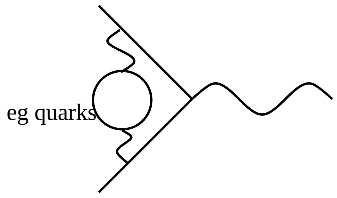
which gives a contribution to the vertex Feynman rule of the form

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2) \tag{82}$$

$F_1$ is the divergent renormalization of the electric charge. $F_2$ though, which is a contribution to the magnetic moment interaction, is finite. A long calculation gives

$$F_2 = \frac{\alpha}{2\pi} = 0.00232 \tag{83}$$

which is even more impressive. At higher order there are many diagrams to consider, and UV divergences enter and must be renormalized. Virtual loops such as




probe the physics of quarks and even potentially particles that have not been discovered on-shell yet. To date the computation for the electron has been completed to order $\alpha^4$ and matches experiment to 8 significant figures. QED is therefore one of the most stringently tested theories we have ever known.