Quantum Electrodynamics

Question 1:

Show that the $\vec{E}$ and $\vec{B}$ fields are invariant to gauge transformations.

Answer 1:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi, \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

The gauge transformations (with an arbitrary choice of $\psi(x^\mu)$) are

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\psi, \qquad \phi \rightarrow \phi - \frac{\partial \psi}{\partial t}$$

So

$$\vec{B} \;\rightarrow\; \vec{\nabla} \times \vec{A} \;+\; \vec{\nabla} \times \vec{\nabla}\psi$$

The second term vanishes since $\vec{\nabla} \times \vec{\nabla}\psi = 0$ for any function ψ. We're left with the original $\vec{B}$.

$$\vec{E} \;\rightarrow\; -\frac{\partial \vec{A}}{\partial t} \;-\; \frac{\partial \vec{\nabla}\psi}{\partial t} \;-\; \vec{\nabla}\phi \;+\; \vec{\nabla}\frac{\partial \psi}{\partial t}$$

The two new terms cancel against each other leaving the original $\vec{E}$.

Question 2:(involved)

Use minimal substitution ($\vec{p} \rightarrow \vec{p} + e\vec{A}$) in the Lagrangian describing a non-relativistic charged particle in a time independent magnetic field and show that the Euler Lagrange equations are the ones you would expect.

Answer 2:

In a time independent magnetic field one has the electric potential $\phi = 0$ and the vector potential $\vec{A}$ is time independent ($\vec{B} = \vec{\nabla} \times \vec{A}$).

The Lagrangian of a free particle is just $L = \frac{1}{2}\dot{\vec{x}}^2 = p^2/2m$. Minimal substitution forces

$$\vec{p} \;\rightarrow\; \vec{p} + e\vec{A}$$

and hence

$$L = \frac{1}{2}\dot{\vec{x}}^2 + q(\dot{\vec{x}}.\vec{A})$$

I've dropped the A^2 term because it won't enter the Euler Lagrange equations.

The Euler Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) - \frac{\partial L}{\partial \vec{x}} = 0$$

or

$$\frac{d}{dt}(m\dot{\vec{x}} + q\vec{A}) - \vec{\nabla}(q\dot{\vec{x}}.\vec{A}) = 0$$

To see this is the equation we want we must first be careful about the time dependence of $\vec{A}$. Of course it doesn't explicitly depend on time (ie drop $\partial\vec{A}/\partial t$), but even if it's constant the particle, as it moves, will see a time variation of the field. This is accounted for using the chain rule

$$\frac{d}{dt} \;=\; \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} \;=\; \frac{\partial}{\partial t} + \dot{\vec{x}}.\vec{\nabla}$$

So our equation of motion is

$$\frac{d\vec{p}}{dt} + q\dot{\vec{x}}.\vec{\nabla}\vec{A} - q\vec{\nabla}(\dot{\vec{x}}.\vec{A}) = 0$$

Next we use the identity

$$\dot{\vec{x}} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\dot{\vec{x}}.\vec{A}) - (\dot{\vec{x}}.\vec{\nabla})\vec{A}$$

We have

$$\frac{d\vec{p}}{dt} = q\dot{\vec{x}} \times \vec{\nabla} \times \vec{A}$$

Rewriting in terms of $\vec{B}$ we have

$$\frac{d\vec{p}}{dt} = q\dot{\vec{x}} \times \vec{B}$$

which hopefully you recognize!

Question 3:(involved)

Starting from eqn (41), which you may assume holds for the Klein Gordon equation, compute the leading order Feynman rules for a spinless, charged particle scattering with a photon.

Answer 3:

The KG equation, after minimal substitution may be written as

$$(\Box + m^2)\phi \ + \ \delta V \phi = 0$$

where

$$\delta V = ie(\partial_\mu A^\mu + A_\mu \partial^\mu) + \mathcal{O}(e^2)$$

I've dropped the A^2 term since it's sub-leading in the e expansion.

For the particle to scatter from a state a to c via a photon interaction we have from eqn (41)

$$\begin{aligned} \kappa_{ca} &= -i \int \phi_c^* \delta V \phi_a d^4x \\ &= -i \int \phi_c^* ie(\partial_\mu A^\mu + A_\mu \partial^\mu)\phi_a d^4x \\ &= \int e\left[\phi_c^* \partial_\mu \phi_a - (\partial_\mu \phi_c^*)\phi_a\right] A^\mu d^4x \end{aligned}$$

I've integrated by parts here and thrown away the surface term at infinite x (one assumes that the value of the A^μ field out there is irrelevant to physics here, so zero!). Note this interaction is again of the form $J^\mu A_\mu$ but with the KG charge current, J^μ.

We shall assume that outside the interaction region the particles are described by solutions of the free KG equation - $\phi = Ne^{-ip.x}$. We have

$$\kappa_{ca} = -i \int eN_a N_c^* (p_a^\mu + p_c^\mu) e^{i(p_c - p_a).x} A_\mu d^4x$$

Next we compute the A^μ field produced by another particle scattering from a state b to a state d

$$\Box A^\mu \ = \ J_{bd}^\mu = eN_b N_d^* (p_b + p_d)^\mu e^{i(p_d - p_b).x}$$

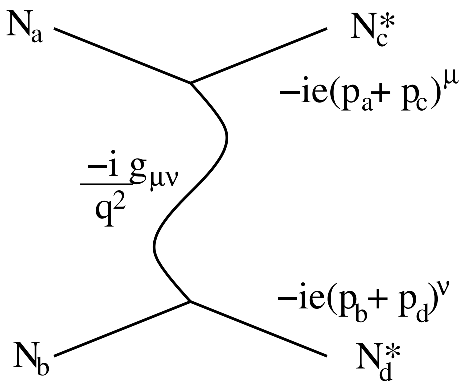
The solution is

$$A^\mu = -\frac{1}{q^2} J_{bd}^\mu, \qquad q^2 = (p_b - p_d)^2$$

We arrive at the expression

$$\kappa_{fi} = N_a N_b N_c^* N_d^* \int e^{i(p_c + p_d - p_a - p_b).x} d^4x ie(p_a + p_c)^\mu \left[\frac{-ig_{\mu\nu}}{q^2}\right] ie(p_b + p_d)^\nu$$

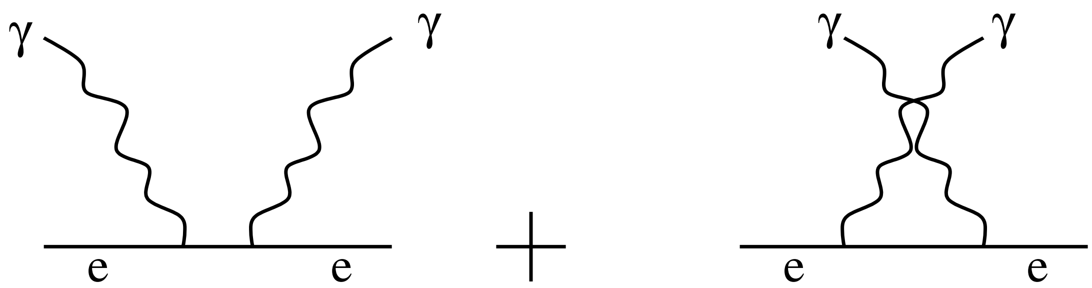
We can associate this with the Feynman diagram rules:



Question 4:

Draw the two Feynman diagrams appropriate to Compton scattering.

Answer 4:



Question 5:

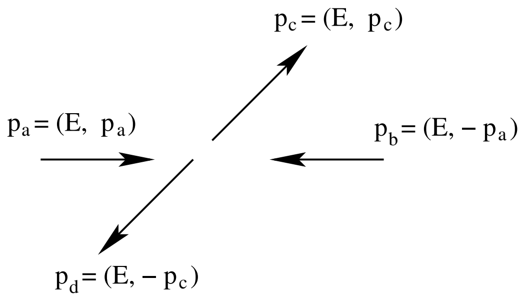
Show that for two body scattering of particles of equal mass m

$$s \geq 4m^2, \quad t \leq 0, \quad u \leq 0$$

Hint: since all variables are Lorentz invariant work in the CoM frame.

Answer 5:

In the centre of mass frame we can draw the event as



where $|\vec{p}_a| = |\vec{p}_c|$. Now we can compute the frame invariant quantities using four-vector multiplication rules

$$s = (p_a + p_c)^2 = (2E, \vec{0})^2 = 4E^2 \geq 4m^2$$

$$t = (p_a - p_c)^2 = (0, \vec{p}_a - \vec{p}_c)^2 = 2|\vec{p}_a|^2(\cos\theta - 1) \leq 0$$

$$u = (p_a - p_d)^2 = (0, \vec{p}_a - \vec{p}_d)^2 = 2|\vec{p}_a|^2(\cos\theta - 1) \leq 0$$

Question 6:

Prove the Gordon Decomposition.

Answer 6:

It's easiest to start with

$$\frac{1}{2m}\bar{u}_f\left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]u_i, \qquad \sigma^{\mu\nu}=\frac{i}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$$

Let's just do the terms with p_i^μ in. Use the Clifford algebra to write $\gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\nu + 2g^{\mu\nu}$ so we have

$$\frac{1}{2m}\bar{u}_f\left[p_i^\mu + \gamma^\mu\gamma^\nu - p_i^\mu\right]u_i$$

The first and last term cancel. Now use the Dirac equation after substituting in a free solution which becomes $p_i^\nu\gamma_\nu u_i = mu_i$ we arrive at

$$\frac{1}{2}\bar{u}_f\gamma^\mu u_i$$

Now go back and repeat this process on the p_f^μ terms... one gets the same factor out and we thus find by adding

$$\frac{1}{2m}\bar{u}_f\left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\right]u_i = \bar{u}_f\gamma^\mu u_i$$