On the Union Bounds of Self-Concatenated Convolutional Codes

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***Abstract*—In this contribution, the union bounds of self-concatenated convolutional codes (SECCCs) are derived for communications over both Additive White Gaussian Noise (AWGN) and uncorrelated Rayleigh fading channels. The truncated union bounds of SECCCs are very useful for studying the corresponding bit error ratio (BER) floors. Based on the truncated union bounds, various SECCCs can be designed for a desired BER floor without the need of time-consuming Monte-Carlo simulations.**

***Index Terms*—Code design, self-concatenated convolutional codes, union bound.**

I. Introduction

Turbo codes based on parallel-concatenated convolutional codes (PCCCs) using two or more constituent Convolutional Codes (CCs) were proposed in [1]. The discovery of turbo codes was a breakthrough in coding theory, because they are capable of operating near the Shannon limit [2]. Serially-concatenated convolutional codes (SCCCs) [3] have been shown to yield a performance comparable, and in some cases superior, to turbo codes. Self-concatenated convolutional codes (SECCCs) proposed by Benedetto *et al.* [4] and Loeliger [5] constitute another attractive iterative detection aided code-family.

SECCCs exhibit a low complexity, since they invoke only a single encoder and a single decoder. Near-capacity SECCCs have been designed in [6] based on extrinsic information transfer (EXIT) charts [7]. All concatenated coding schemes including SECCCs tend to exhibit a Bit Error Ratio (BER) floor in the medium to high signal-to-noise ratio (SNR) region. However, the BER floor of SECCCs has not been analyzed in the literature. While the EXIT chart analysis is only accurate when a sufficiently long interleaver is used, the BER floor analysis using truncated union bound is valid for arbitrary interleaver lengths. Hence, the BER floor analysis is important for code design. More specifically, the union bound constitutes a useful code design technique [8]–[10], which was also employed for the design of antenna selection schemes [11].

In this contribution, we first study the similarities and differences between PCCCs and SECCCs in Section II. Then, we highlight the union bound derivation for conventional CCs in Section III, before we derive the union bound of SECCCs in Section IV. The union bound derived is then compared to our simulation results in Section V and our conclusions are offered in Section VI.

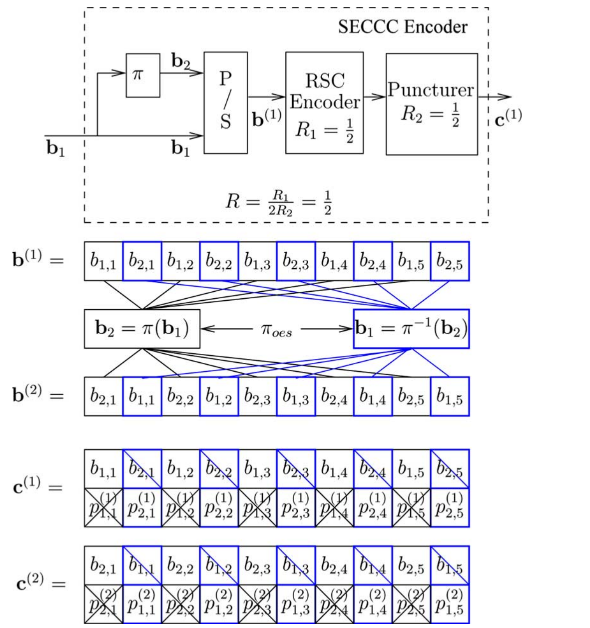

Fig. 1. Schematic of the SECCC encoder. The notations $\mathbf{b}^{(1)}$ and $\mathbf{b}^{(2)}$ denote the information sequences of the hypothetical upper and lower component encoder, respectively, while the puncturer output sequences of the hypothetical upper and lower component encoder are denoted as $\mathbf{c}^{(1)}$ and $\mathbf{c}^{(2)}$, respectively.

II. System Model

The schematic of the SECCC encoder employing a $R_1 = 1/2$ recursive systematic convolutional (RSC) encoder and a $R_2 = 1/2$ puncturer is shown in Fig. 1. As seen from Fig. 1, the bit sequence $\mathbf{b}_2 = [b_{2,1}\ b_{2,2}\ b_{2,3} \ldots]$ is simply the interleaved version of the original bit sequence $\mathbf{b}_1 = [b_{1,1}\ b_{1,2}\ b_{1,3} \ldots]$. After the parallel-to-serial (P/S) conversion, we can compute the information sequence of the hypothetical upper SECCC component code as $\mathbf{b}^{(1)} = [b_{1,1}\ b_{2,1}\ b_{1,2}\ b_{2,2} \ldots]$. Interestingly, we can view the information sequence of the hypothetical lower

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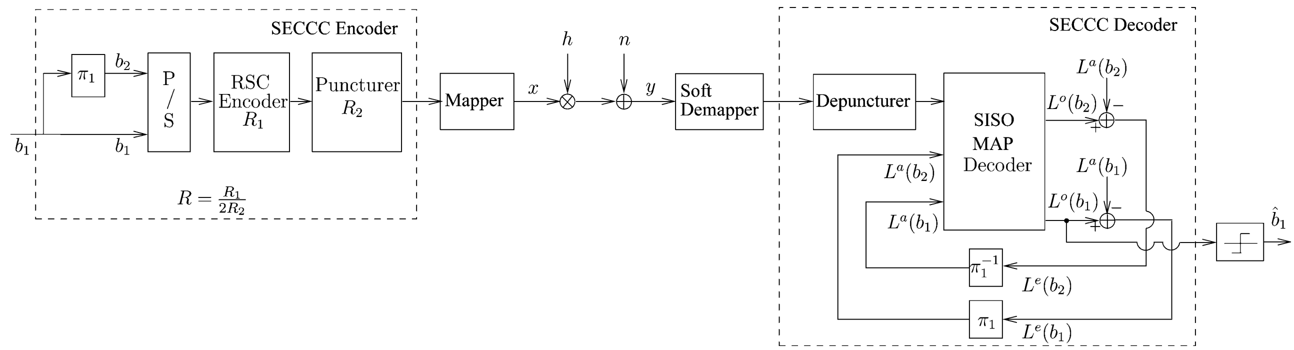

Fig. 2. Schematic of the SECCC encoder and decoder when communicating over Rayleigh fading channels. The notation $L(b)$ denotes the LLR of bit b and the the supercripts a, o and e denote a priori, a posteriori and extrinsic nature of the LLR, respectively.

SECCC component code $\mathbf{b}^{(2)}$ as the interleaved version of $\mathbf{b}^{(1)}$ using an odd–even separation (OES) based interleaver π_{oes}. More explicitly, the OES interleaver consists of two component interleavers, where the odd position of the bit sequence is permuted based on the mapping of $\pi_o = \pi$, while the even position of the bit sequence is permuted based on the inverse of the mapping π, namely on $\pi_e = \pi^{-1}$.

We apply a puncturer that removes the interleaved bit sequence $\mathbf{b}_2$ as well as all parity bits corresponding to the bit sequence $\mathbf{b}_1$ in order to yield the output sequence $\mathbf{c}^{(1)}$, as shown in Fig. 1. The resultant puncturing rate is given by $R_2 = 1/2$ and the SECCC output sequence $\mathbf{c}^{(1)}$ consists of only the input bit sequence $\mathbf{b}_1$ as well as the parity bit sequence corresponding to $\mathbf{b}_2$, as shown in Fig. 1. The SECCC encoder consists of both the rate-R_1 RSC encoder and the rate-R_2 puncturer. Hence, the coding rate of the SECCC encoder, as shown in Fig. 1, is given by $R = R_1/(2R_2) = 1/2$.

Although $\mathbf{b}_2$ is punctured from $\mathbf{c}^{(1)}$, we can obtain the Log-Likelihood Ratio (LLR) of the bits in $\mathbf{b}_2$ by interleaving the LLRs associated with the bits in $\mathbf{b}^{(1)}$ obtained from the MAP decoder, as shown in Fig. 2. Hence, the output sequence $\mathbf{c}^{(1)}$ seen in Fig. 1 is similar to that of the upper component encoder of a turbo code, where all parity bits corresponding to the odd-position information bits are punctured. Similarly, all parity bits corresponding to the odd-position information bits at the output sequence of the hypothetical lower component code $\mathbf{c}^{(2)}$, as seen in Fig. 1, are punctured.

Based on these observations, we are able to compute the union bound of SECCCs, as detailed in Section IV.

III. Union Bounds of Convolutional Codes

The pair-wise error probability (PWEP) is defined as the probability that the modulated symbol sequence $\mathbf{x} = [x_1\, x_2\, \ldots]$ is wrongly decoded as another modulated symbol sequence $\hat{\mathbf{x}} = [\hat{x}_1\, \hat{x}_2\, \ldots]$. The PWEP, which depends on both the modulation scheme as well as on the code structure and the communication channel, can be expressed as [8]

$$P(\mathbf{x} \rightarrow \hat{\mathbf{x}}) = \mathrm{Q}\left(\sqrt{\frac{\gamma}{2} d^2(\mathbf{x}, \hat{\mathbf{x}})}\right) \quad (1)$$

where γ is the SNR, while $d^2(\mathbf{x}, \hat{\mathbf{x}})$ is the squared *Euclidean distance* between the modulated symbol sequences $\mathbf{x}$ and $\hat{\mathbf{x}}$, when communicating over AWGN channel, which is given by

$$d^2(\mathbf{x}, \hat{\mathbf{x}}) = \sum_{t \in \eta} |x_t - \hat{x}_t|^2 \quad (2)$$

where η represents the set of indices t satisfying the condition of $x_t \neq \hat{x}_t$. The number of elements in the set η is given by $\Delta_H = \Delta_H(\mathbf{x}, \hat{\mathbf{x}})$, which quantifies the number of erroneous modulated symbols in the sequence $\hat{\mathbf{x}}$, when compared to the correct sequence $\mathbf{x}$.When communicating over uncorrelated Rayleigh fading channels, the PWEP can be shown to be [12]

$$P(\mathbf{x} \rightarrow \hat{\mathbf{x}}) \leq \frac{1}{2} \prod_{t \in \eta} \left(1 + \frac{\gamma}{4} |x_t - \hat{x}_t|^2\right)^{-1}. \quad (3)$$

In this contribution, we will derive the union bound of SECCCs based on BPSK modulation. Note that when BPSK modulation is employed, we have $|x_t - \hat{x}_t|^2 = 4$, whenever $x_t \neq \hat{x}_t$ since $\{x_t, \hat{x}_t\} = \{\pm 1\}$. Based on this simplification, the PWEP for the AWGN channel can be expressed from (1) as

$$P(\mathbf{x} \rightarrow \hat{\mathbf{x}}) = \mathrm{Q}\left(\sqrt{2\gamma\Delta_H}\right). \quad (4)$$

Similarly, the PWEP for the uncorrelated Rayleigh fading channel can be simplified from (3) as

$$P(\mathbf{x} \rightarrow \hat{\mathbf{x}}) \leq \frac{1}{2}(1 + \gamma)^{-\Delta_H} \quad (5)$$

where Δ_H is also referred to as the *effective Hamming distance*, which quantifies the diversity order of the code.

The union bound of the average BER of a coding scheme can be expressed as [8]

$$P_b \leq \frac{1}{k} \sum_{\Delta_H} B_{\Delta_H} P(\mathbf{x} \rightarrow \hat{\mathbf{x}}) \quad (6)$$

where k is the number of information bits per n-bit coded symbol and B_{Δ_H} is the distance spectrum of the code, given by

$$B_{\Delta_H} = \sum_{w} \frac{w}{N} \cdot A_{w,\delta}; \text{ for } \Delta_H = w + \delta \quad (7)$$

where w is the information weight denoting the number of erroneous information bits in an encoded sequence and δ is the parity weight quantifying the number of erroneous parity bits in an encoded sequence. More explicitly, $A_{w,\delta}$ is the two-dimensional Weight Enumerating Function (WEF), quantifying the average number of sequence error events having an information weight of w and a parity weight of δ. Hence, the Hamming distance is given by $\Delta_H = w + \delta$.

IV. Union Bounds of SECCCs

The WEF of SECCCs can be expressed as

$$A_{w,\delta} = A^{(1)}_{2w,\delta^{(1)}} \cdot A^{(2)}_{2w,\delta^{(2)}} \cdot P^{N,w}_{\pi} \tag{8}$$

where $A^{(1)}_{2w,\delta^{(1)}}$ and $A^{(2)}_{2w,\delta^{(2)}}$ are the WEFs of the hypothetical upper and lower component codes, respectively, while the effective parity weight of an SECCC is given by

$$\delta = \delta^{(1)} + \delta^{(2)} \tag{9}$$

where $\delta^{(1)}$ and $\delta^{(2)}$ are the parity weights of the hypothetical upper and lower component codes, respectively. As we can see from Fig. 1, the information sequence of the upper component encoder $\mathbf{b}^{(1)}$ consists of the original information sequence $\mathbf{b}_1$ and its interleaved version $\mathbf{b}_2$. Hence, if the original information sequence $\mathbf{b}_1$ has an information weight of w, then the information sequence of the upper component encoder $\mathbf{b}^{(1)}$ will have an information weight of $2w$. The same also applies to the lower component code. Hence, we have $A^{(1)}_{2w,\delta^{(1)}}$ and $A^{(2)}_{2w,\delta^{(2)}}$ in (8).

The term $P^{N,w}_{\pi}$ in (8) denotes the probability of occurrence for all the associated error events having w information bit errors, when employing a self-concatenated bit-interleaver having a length of N bits. The evaluation of $P^{N,w}_{\pi}$ is based on the novel uniform self-interleaver concept, which may be interpreted as the extension of the uniform bit-interleaver concept proposed in [9]. More specifically, a uniform self-interleaver may be partitioned into two bit-interleavers, as defined in Definition 1.

Definition 1: A uniform self-interleaver of length N bits is a probabilistic device, which maps a given input sequence of length N bits having an information weight of w bits into all possible permutations in the odd and even partitions of an equivalent odd-even-separation based interleaver of length $2N$ having an information weight of $2w$, with equal probability of $P^{N,w}_{\pi}$ given by

$$P^{N,w}_{\pi} = P^{N,w} \cdot P^{N,w} \tag{10}$$

where $P^{N,w} = 1/\binom{N}{w}$, which characterizes the traditional N-bit uniform interleaver having an information weight of w bits. If there are w bit errors in the information sequence, then there will be w bit errors in the 'odd' sequence $\mathbf{b}_1$ as well as another w bit errors in the 'even' sequence $\mathbf{b}_2$, since $\mathbf{b}_2$ is simply the interleaved version of the $\mathbf{b}_1$ sequence.

The WEF $A_{w,\delta}$ for an SECCC having a block length of N encoded symbols and a total of M number of trellis states can be calculated as follows. We can define the State Input-Redundancy WEF (SIRWEF) for a block of N SECCC-encoded symbols as

$$\mathbf{A}(N,S,W,Z) = \sum_{w}\sum_{\delta} A_{N,S,w,\delta} \cdot W^{w} Z^{\delta} \tag{11}$$

where $A_{N,S,w,\delta}$ is the number of paths in the trellis entering state S at symbol index N, which have an information weight of w and a parity weight of δ. The notations W and Z represent dummy variables. For each n-bit coded symbol at index t, the term $A_{t,S,w,\delta}$ can be calculated recursively as follows:

$$A_{t,S,w,\delta} = \sum_{S',S:u_t} A_{t-1,S',w',\delta'},\ (1 \le t \le N) \tag{12}$$

where u_t represents the specific k-bit input symbol that triggers the transition from state S' at index $(t-1)$ to state S at index t, while the terms w and δ can be formulated as

$$w = w' + i(S',S);\ \delta = \delta' + \Phi(S',S) \tag{13}$$

where w' and δ' are the information weight and the parity weight, respectively, of the trellis paths entering state S' at index $(t-1)$. Furthermore, $i(S',S) \in \{0,1,\ldots,k\}$ is the information weight of the k-bit information symbol u_t that triggers the transition from state S' to S and $\Phi(S',S) \in \{0,1,\ldots,n-k\}$ is the parity weight between $\hat{c}_t$ and c_t, where $\hat{c}_t$ is the encoded n-bit symbol corresponding to the trellis branch in the transition from state S' to S and c_t is the actual encoded n-bit symbol at index t. Again, all the parity bits in $\{c_t\}$ (or $\{\hat{c}_t\}$) corresponding to the odd-position information bits are punctured. Note that the parity weight contribution corresponding to a punctured parity bit equals to zero.

Let the encoding process commence from state 0 at index 0 and terminate at any of the M possible states at index N. Then the WEF used in (7) is given by

$$A_{w,\delta} = \sum_{S} A_{N,S,w,\delta}. \tag{14}$$

Note that for linear codes [13] the distance profile of the code is independent of which particular encoded symbol sequence is considered to be the correct one. Hence, for the sake of simplicity, we can assume that the all-zero encoded symbol sequence is transmitted.

Based on all the above equations, the union bound of an SECCC employing BPSK modulation can be shown to be

$$P_b \le \sum_{\Delta_H}\sum_{w} \frac{A^{(1)}_{2w,\delta^{(1)}} \cdot A^{(2)}_{2w,\delta^{(2)}}}{\binom{N}{w}\cdot\binom{N}{w}} \cdot \frac{w \cdot \mathrm{Q}\left(\sqrt{2\gamma\Delta_H}\right)}{kN} \tag{15}$$

when communicating over AWGN channels and

$$P_b \le \sum_{\Delta_H}\sum_{w} \frac{A^{(1)}_{2w,\delta^{(1)}} \cdot A^{(2)}_{2w,\delta^{(2)}}}{\binom{N}{w}\cdot\binom{N}{w}} \cdot \frac{w \cdot (1+\gamma)^{-\Delta_H}}{2kN} \tag{16}$$

when communicating over uncorrelated Rayleigh fading channels, where $\Delta_H = w + \delta^{(1)} + \delta^{(2)}$.

V. Results and Discussions

Let us now compare the BER performance of CCs and SECCCs to their union bounds truncated at a maximum Hamming distance of $\Delta_{H\,\max} = w_{\max} + \delta_{\max} = 20$, where the maximum information and parity weights considered are $w_{\max} = 10$ and $\delta_{\max} = 10$, respectively. Figs. 3 and 4 shows the BERs of our simulations and bounds of the CCs and

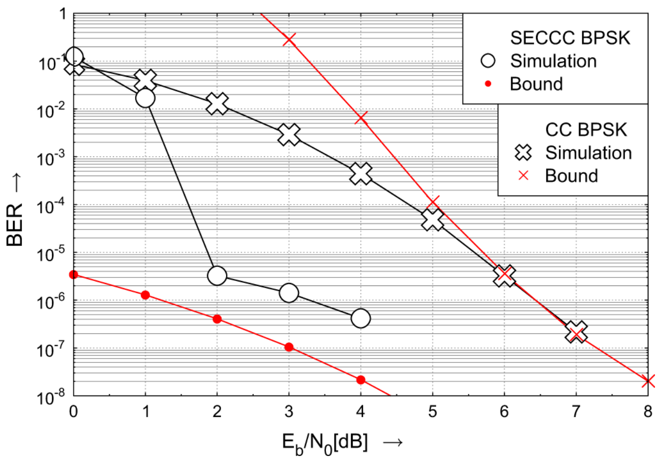

Fig. 3. Simulations and truncated union bounds of BPSK-assisted CC and SECCC, when communicating over AWGN channels. The union bounds are truncated at a maximum Hamming distance of $\Delta_{H\max} = 20$. The SECCC employs an interleaver of length 12000 bits and 8 decoding iterations.

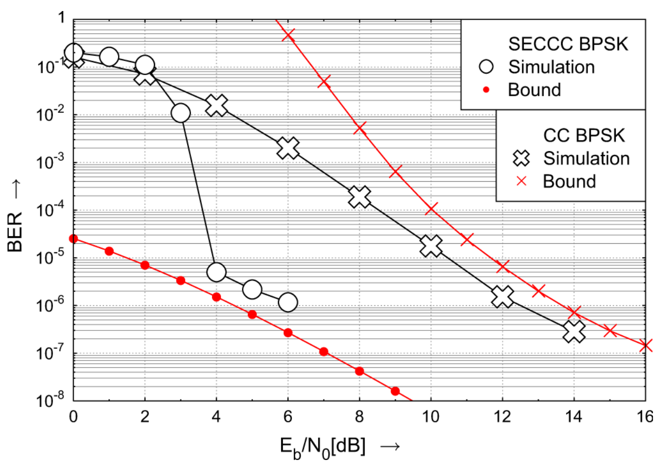

Fig. 4. Simulations and truncated union bounds of BPSK-assisted CC and SECCC, when communicating over uncorrelated Rayleigh fading channels. The union bounds are truncated at a maximum Hamming distance of $\Delta_{H\max} = 20$. The SECCC employs an interleaver of length 12 000 bits and eight decoding iterations.

SECCCs employing BPSK modulation, when communicating over both AWGN and uncorrelated Rayleigh fading channels. Both the CC and SECCC employ an RSC code based on a generator polynomial of $G = [13\ 15]$ expressed in octal format.

As shown in Figs. 3 and 4, the truncated union bound quantifies the BER floor of SECCCs quite accurately. Hence, we can design various SECCCs having various desired BER floors using the proposed truncated union bound.

VI. Conclusions

A useful union bound has been derived for BPSK-based SECCCs, when communicating over both AWGN and uncorrelated Rayleigh fading channels. The union bound can be truncated in order to conveniently analyze the BER floor of SECCCs. This union bound can be used together with EXIT charts in order to design near-capacity SECCCs operating at a given desired BER floor. It can also be extended to high-order modulation schemes for designing bandwidth efficient SECCCs.

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