Multiple-expectation Systems

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Outline

1. Motivation
   - Extension to Probabilistic B
   - Background

2. Our Results/Contribution
   - Multiple Probabilistic Specification Substitutions
   - Fundamental Theorem
   - Case Study: Duelling Cowboys
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1 Motivation
   - Extension to Probabilistic B
   - Background

2 Our Results/Contribution
   - Multiple Probabilistic Specification Substitutions
   - Fundamental Theorem
   - Case Study: Duelling Cowboys
To extend the scope of probabilistic B (pB) to cover systems with multiple probabilistic properties;

- Need to introduce *multiple probabilistic specification substitution*;
- Investigate the new substitution in the framework of *layered developments*.
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Extending probabilistic B

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- Need to introduce *multiple probabilistic specification substitution*;
- Investigate the new substitution in the framework of *layered developments*. 
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How \( pGSL \) extends \( GSL \)

**Expectations replace predicates**

*Predicates* (functions from state to Boolean) are widened to *Expectations* (functions from state to non-negative real).

- For consistency with Boolean logic, we use embedded predicates, \( \langle \text{false} \rangle = 0 \), and \( \langle \text{true} \rangle = 1 \).
- Notationally, we have kept predicates as much as possible.
How $pGSL$ extends $GSL$

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How \textit{pGSL} extends \textit{GSL}

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- Notationally, we have kept predicates as much as possible.
Summary

\[ x : = E \] \text{exp}

The expectation obtained after replacing all free occurrences of \( x \) in \( \text{exp} \) by \( E \)

\[ \text{[skip]} \text{exp} \]
\[ \text{[prog\(_1\) \( p \oplus \) prog\(_2\)] \text{exp} \]
\[ \text{prog\(_1\) \( \sqsubseteq \) prog\(_2\) \]
\[ \text{[prog\(_1\) \( \mid \) prog\(_2\)] \text{exp} \]
\[ \text{[@y \cdot pred \( \Longrightarrow \) prog] \text{exp} \]

\[ p \times \text{[prog\(_1\)] \text{exp} \]
\[ + (1 - p) \times \text{[prog\(_2\)] \text{exp} \]
\[ \text{[prog\(_1\)] \text{exp} \Rightarrow \text{[prog\(_2\)] \text{exp} \]
\[ \text{[prog\(_1\)] \text{exp} \text{ min} \text{[prog\(_2\)] \text{exp} \]
\[ \text{min} (y) \cdot (\text{pred} \mid \text{[prog]} \text{exp}) \]

Probabilistic generalised substitution language
Probabilistic generalised substitution language

Summary

\[ x : = E \] \( \text{exp} \)

The expectation obtained after replacing all free occurrences of \( x \) in \( \text{exp} \) by \( E \)

\[ \text{[skip]} \text{exp} \]

\[ \text{[prog}_1 \text{ } p \oplus \text{prog}_2 \text{]} \text{exp} \]

\[ \text{prog}_1 \sqsubseteq \text{prog}_2 \]

\[ \text{[prog}_1 \parallel \text{prog}_2 \text{]} \text{exp} \]

\[ \text{[@}y \cdot \text{pred} \implies \text{prog} \text{]} \text{exp} \]

\[ p \times [\text{prog}_1] \text{exp} \]

\[ + (1 - p) \times [\text{prog}_2] \text{exp} \]

\[ [\text{prog}_1] \text{exp} \Rightarrow [\text{prog}_2] \text{exp} \]

\[ [\text{prog}_1] \text{exp} \min [\text{prog}_2] \text{exp} \]

\[ \min (y) \cdot (\text{pred} \mid [\text{prog}] \text{exp}) \]
Probabilistic generalised substitution language

Summary

\[
\begin{align*}
[x := E]\exp & \quad \text{The expectation obtained after replacing all free occurrences of } x \text{ in } \exp \text{ by } E \\
[skip]\exp & \quad \exp \\
[\text{prog}_1, p \oplus \text{prog}_2]\exp & \quad p \times [\text{prog}_1]\exp \\
& \quad + (1-p) \times [\text{prog}_2]\exp \\
\text{prog}_1 \sqsubseteq \text{prog}_2 & \quad [\text{prog}_1]\exp \Rightarrow [\text{prog}_2]\exp \\
[\text{prog}_1 \parallel \text{prog}_2]\exp & \quad [\text{prog}_1]\exp \text{ min } [\text{prog}_2]\exp \\
[@y \cdot \text{pred} \Rightarrow \text{prog}]\exp & \quad \text{min } (y) \cdot (\text{pred} \mid [\text{prog}]\exp)
\end{align*}
\]
Probabilistic generalised substitution language

Summary

\[ [x : = E] \exp \]

The expectation obtained after replacing all free occurrences of \( x \) in \( \exp \) by \( E \)

\[ [\text{skip}] \exp \]

\[ [\text{prog}_1 \oplus \text{prog}_2] \exp \]

\[ \text{prog}_1 \sqsubseteq \text{prog}_2 \]

\[ [\text{prog}_1 || \text{prog}_2] \exp \]

\[ [@y \cdot \text{pred} \Rightarrow \text{prog}] \exp \]

\[ p \times [\text{prog}_1] \exp \]

\[ + (1-p) \times [\text{prog}_2] \exp \]

\[ [\text{prog}_1] \exp \Rightarrow [\text{prog}_2] \exp \]

\[ [\text{prog}_1] \exp \text{ min} [\text{prog}_2] \exp \]

\[ \text{min} (y) \cdot (\text{pred} \mid [\text{prog}] \exp) \]
Probabilistic generalised substitution language

**Summary**

\[ x \overset{=}{=} E \exp \]

The expectation obtained after replacing all free occurrences of \( x \) in \( \exp \) by \( E \)

\[ \text{[skip]}\exp \quad \text{[exp]} \]

\[ \text{[prog}_1 \oplus \text{prog}_2\text{]}\exp \]

\[ p \times \text{[prog}_1\text{]}\exp + (1-p) \times \text{[prog}_2\text{]}\exp \]

\[ \text{prog}_1 \sqsubseteq \text{prog}_2 \]

\[ \text{[prog}_1 \parallel \text{prog}_2\text{]}\exp \]

\[ \text{[prog}_1\text{]}\exp \min \text{[prog}_2\text{]}\exp \]

\[ \text{[@y \cdot pred} \Rightarrow \text{prog}\text{]}\exp \]

\[ \text{min} (y) \cdot (\text{pred} \mid \text{[prog]}\exp) \]
(Single) Probabilistic specification substitution

Syntax

\( v : \{ A , B \} \), where \( A \) and \( B \) are expectations over state \( x \).

- \( v \subseteq x \)
- \( B \) can refer to the original state by using subscripted variables \( x_0 \).

The expected value of \( B \) over the set of final distributions is at least the expected value of \( A \) over the initial distribution.

Semantics

\[
[v : \{ A , B \}] C \triangleq A \times [x_0 : = x] \left( \cap x \cdot \left( \frac{C}{B \times \langle w = w_0 \rangle} \right) \right)
\]

(\( w \) is the set of unchanged variables, i.e. \( x - v \)).
(Similar work can be seen in White[1996] and Ying[2003])
(Single) Probabilistic specification substitution

Syntax

\[ v : \{ A , B \} , \text{ where } A \text{ and } B \text{ are expectations over state } x. \]

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- \( B \) can refer to the original state by using subscripted variables \( x_0 \).

The expected value of \( B \) over the set of final distributions is at least the expected value of \( A \) over the initial distribution.

Semantics

\[ [v : \{ A , B \}] C \equiv A \times [x_0 : = x] \left( \forall x \cdot \left( C \middle/ B \times \langle w = w_0 \rangle \right) \right) \]

\( w \) is the set of unchanged variables, i.e. \( x - v \).

(Similar work can be seen in White[1996] and Ying[2003])
Fundamental theorem

Probabilistic Theorem

Assume that $\text{prog}_1 \equiv \nu : \{A, B\}$. For any program $\text{prog}_2$,

$$\text{prog}_1 \subseteq \text{prog}_2$$

if and only if

$$A \Rightarrow [x_0 := x] [\text{prog}_2] B^w,$$

where $B^w \equiv B \times \langle w = w_0 \rangle$. 
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For $i \in (1..n)$, let $p_i$ be a probabilistic expression over the state satisfying
\[
\sum_{i=1}^{n} p_i \leq 1 ;
\]  
(1)

Let $S_i$ be a probabilistic substitution. The multi-way probabilistic choice is defined as follows:
\[
E \equiv p_1 \times [S_1] E + p_2 \times [S_2] E + \cdots + p_n \times [S_n] E .
\]  
(2)

where $E$ is an arbitrary expectation of the state.
For $i \in (1..n)$, let $p_i$ be a probabilistic expression over the state $x$ and free from $x_0$ and satisfying:

$$\sum_{i=1}^{n} p_i \leq 1 ;$$  \hfill (3)

let $Q_i$ be predicates defined over $x_0, \nu$ (where $\nu$ is a subset of $x$) and satisfying, for all $Q_i$, that we have

$$\forall x_0 \cdot (\exists \nu \cdot Q_i) .$$  \hfill (4)
**Semantics**

Let $p_0 = 1 - \sum_{i=1}^{n} p_i$, we define

$$v : \begin{cases} \{p_1, \langle Q_1 \rangle\} \\ \{p_2, \langle Q_2 \rangle\} \\ \vdots \\ \{p_n, \langle Q_n \rangle\} \end{cases} \cong \begin{cases} (v : \{1, \langle Q_1 \rangle\}) @ p_1 \\ (v : \{1, \langle Q_2 \rangle\}) @ p_2 \\ \vdots \\ (v : \{1, \langle Q_n \rangle\}) @ p_n \\ (x : \{1, 1\}) @ p_0 \end{cases} \quad (5)$$
Let $p_0 = 1 - \sum_{i=1}^{n} p_i$, we define

$$v : \begin{cases} \{p_1, \langle Q_1 \rangle \} \\
\{p_2, \langle Q_2 \rangle \} \\
\text{...} \\
\{p_n, \langle Q_n \rangle \} \end{cases} \equiv \begin{cases} (v : \{1, \langle Q_1 \rangle \}) \@ p_1 \\
(v : \{1, \langle Q_2 \rangle \}) \@ p_2 \\
\text{...} \\
(v : \{1, \langle Q_n \rangle \}) \@ p_n \\
(x : \{1, 1\}) \@ p_0 \end{cases} \quad (5)$$
Examples

A fair coin

\[ S_1 \equiv c : \{ \frac{1}{2}, \langle c = H \rangle \} \]
\[ \{ \frac{1}{2}, \langle c = T \rangle \} \] (6)

A non-deterministic coin:

A coin which guarantees to return Heads at least 1/3 of the time and Tails at least 1/3 of the time.

\[ S_2 \equiv c : \{ \frac{1}{3}, \langle c = H \rangle \} \]
\[ \{ \frac{1}{3}, \langle c = T \rangle \} \] (7)
Motivation

Our Results/Contribution

Summary

Multiple Probabilistic Specification Substitutions

Examples

A fair coin

\[ S_1 \triangleq c : \begin{cases} \frac{1}{2}, \langle c = H \rangle \\ \frac{1}{2}, \langle c = T \rangle \end{cases} \]  

(6)

A non-deterministic coin:

A coin which guarantees to return Heads at least 1/3 of the time and Tails at least 1/3 of the time.

\[ S_2 \triangleq c : \begin{cases} \frac{1}{3}, \langle c = H \rangle \\ \frac{1}{3}, \langle c = T \rangle \end{cases} \]  

(7)
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We consider a special set of multiple probabilistic specification substitutions where for any pair $Q_i$ and $Q_j$, where $i \neq j$, we have

$$Q_i \land Q_j = false,$$

(8)

**Probabilistic Theorem**

*For all programs $T$, if*

$$(x : \{1 , 1\}) \subseteq T \text{ and } (v : \{p_i , \langle Q_i \rangle\}) \subseteq T, \text{ for all } i \in (1..n),$$

(9)

then we have

$$v : \{p_1, \langle Q_1 \rangle\} \bigcup \{p_2, \langle Q_2 \rangle\} \bigcup \ldots \bigcup \{p_n, \langle Q_n \rangle\} \subseteq T.$$  

(10)

(11)
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Case Study: Duelling Cowboys

Two cowboys

Conditions

There are two cowboys X and Y fighting a duel. They take turns to shoot at each other.

- The probability for X to hit his opponent is \( \frac{2}{3} \).
- The probability for Y to hit his opponent is \( \frac{1}{2} \).
- Assuming that X has the advantage of shooting first.

Question?

What are the guaranteed survival probabilities for both cowboys?
Case Study: Duelling Cowboys

Two cowboys

Conditions

There are two cowboys $X$ and $Y$ fighting a duel. They take turns to shoot at each other.

- The probability for $X$ to hit his opponent is $\frac{2}{3}$.
- The probability for $Y$ to hit his opponent is $\frac{1}{2}$.
- Assuming that $X$ has the advantage of shooting first.

Question?

What are the guaranteed survival probabilities for both cowboys?
Formal specification

Let $p_X$ and $p_Y$ be the survival probability for $X$ and $Y$, respectively. Let $s$ be the cowboy which survives the duelling.

$$s \leftarrow \text{TwoCowboyXYSpec} \quad \equiv \quad s : \quad \{p_X, \langle s = X \rangle \}$$

$$\{p_Y, \langle s = Y \rangle \}$$
Implementation

\[ s \leftarrow \text{TwoCowboyXYImp} \equiv \]

\[ \text{VAR } t, n \text{ IN} \]
\[ t : = X; s : = X; n : = 2; \]
\[ \text{WHILE } n = 2 \text{ DO} \]
\[ \text{IF } t = X \text{ THEN} \]
\[ (s : = X; n : = 1) \quad \text{END} \]
\[ \text{ELSE} \]
\[ (s : = Y; n : = 1) \quad \text{END} \]
\[ \text{EXPECTATIONS } \cdots \]
\[ \text{END} \]
\[ \text{END} \]
**Implementation**

\[ s \leftarrow \text{TwoCowboyXYImp} \sqsubseteq \]

VAR \( t, n \) IN

\[ t : = X ; s : = X ; n : = 2 ; \] // init

WHILE \( n = 2 \) DO

IF \( t = X \) THEN

\((s : = X ; n : = 1) \oplus \frac{2}{3} t : = Y\)

ELSE

\((s : = Y ; n : = 1) \oplus \frac{1}{2} t : = X\)

END

EXPECTATIONS \cdots

END

END
Implementation

\[ s \leftarrow \text{TwoCowboyXYImp} \]
\[ \text{VAR } t, n \text{ IN} \]
\[ t : = X ; s : = X ; n : = 2 ; \]
\[ \text{WHILE } n = 2 \text{ DO} \] // Loop
\[ \text{IF } t = X \text{ THEN} \]
\[ (s : = X ; n : = 1 ) \quad 2 \oplus \quad t : = Y \]
\[ \text{ELSE} \]
\[ (s : = Y ; n : = 1 ) \quad 1 \oplus \quad t : = X \]
\[ \text{END} \]
\[ \text{EXPECTATIONS} \cdots \]
\[ \text{END} \]
\[ \text{END} \]
Case Study: Duelling Cowboys

Implementation

\[ s \leftarrow \text{TwoCowboyXYImp} \equiv \]

\[
\begin{align*}
\text{VAR } t, n \text{ IN} \\
t : = X; s : = X; n : = 2; \\
\text{WHILE } n = 2 \text{ DO} \\
\quad \text{IF } t = X \text{ THEN} \quad \text{body} \\
\quad \quad (s : = X; n : = 1) \quad \frac{2}{3} \oplus t : = Y \\
\quad \text{ELSE} \\
\quad \quad (s : = Y; n : = 1) \quad \frac{1}{2} \oplus t : = X \\
\quad \text{END} \\
\text{EXPECTATIONS} \cdots \\
\text{END} \\
\text{END}
\]
Proof obligations

In order to prove that $\text{TwoCowboyXYSpec} \sqsubseteq \text{TwoCowboyXYImp}$, we have to prove that

$$\text{(s : \{p}_X, \langle s = X\rangle\}) \sqsubseteq \text{TwoCowboyXYImpl} \quad (12)$$

and

$$\text{(s : \{p}_Y, \langle s = Y\rangle\}) \sqsubseteq \text{TwoCowboyXYImpl} \quad . \quad (13)$$

Then we can apply the fundamental theorem for single probabilistic specification substitution for (12) and (13) separately. For (12) we have to prove that

$$p_\text{X} \Rightarrow [\text{init}; \text{Loop}] \langle s = X\rangle . \quad (14)$$
Recall proof rules for probabilistic loops

For a probabilistic loop, such as

\[
\text{loop} \quad \equiv \quad \text{WHILE } G \text{ DO } S \text{ IN Variant I EXPECTATION } E \text{ END .}
\]

then \( A \Rightarrow [\text{init}; \text{loop}] B \) holds if the following satisfies:

\[
\begin{align*}
P1 & \quad A \quad \Rightarrow \quad [\text{init}] E \\
\text{P2} & \quad \langle G \land I \rangle \ast E \quad \Rightarrow \quad [S] E \\
\text{P3} & \quad \langle \neg G \land I \rangle \ast E \quad \Rightarrow \quad B
\end{align*}
\]

(Here, I only concentrate on the maintenance of the expectation \( E \))
For proving (14), we try to “guess” the expectation of the loop by tabulating the probabilities of establishing the post-expectation \( s = X \) after executing one iteration of the loop.

\[
\begin{array}{|c|c|c|}
\hline
\text{ } & n = 2 & s = X \land n = 1 \\
\hline
\text{ } & s = Y \land n = 1 \\
\hline
\text{ } & t = X & 4/5 \\
\hline
\text{ } & 1 \\
\hline
\text{ } & 0 \\
\hline
\text{ } & t = Y & 2/5 \\
\hline
\text{ } & 1 \\
\hline
\text{ } & 0 \\
\hline
\end{array}
\]

We have the expectation of the loop is

\[
E \equiv \langle s = X \land n = 1 \rangle + \langle n = 2 \land t = X \rangle \times \frac{4}{5} + \langle n = 2 \land t = Y \rangle \times \frac{2}{5}. \tag{15}
\]
Apply the proof rule $P1$, we need to prove that

$$p_X \Rightarrow [t: = X; s: = X; n: = 2]E,$$  \hspace{1cm} (16)

which is equivalent to

$$p_X \Rightarrow \frac{4}{5}.$$  \hspace{1cm} (17)

So we can choose $p_X \equiv \frac{4}{5}$, which will be the guaranteed surviving probability for $X$. With similar reasoning, we have $p_Y \equiv \frac{1}{5}$. 

### Case Study: Duelling Cowboys

#### Development in layers

**Three Cowboys**

Assume that we have another cowboy, namely $Z$ with the probability of hitting his opponent is $\frac{1}{3}$. With similar setting, what are the surviving probabilities for the cowboys.

Here, we can use the specification of the two cowboys situation when write the implementation, for example, the case when $Y$ has the turn to shoot can be specified as follows:

\[
\text{IF } t = Y \text{ THEN}
\begin{align*}
    & (s \leftarrow \text{TwoCowboyXY}; n : = 1) && \frac{1}{2} \oplus t : = Z \\
\end{align*}
\]

ELSE \ldots

and the reasoning can be done similarly as in the case for two cowboys.
Abstractly specify and refine probabilistic systems with multiple properties.
Development of these systems can be separated into layers.
When the state is small, the expectation for loops can be found using the tabular method.
For further reading I


